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Solving ill-posed control problems by stabilized finite element methods: an alternative to Tikhonov regularization

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Abstract

Tikhonov regularization is one of the most commonly used methods for the regularization of ill-posed problems. In the setting of finite element solutions of elliptic partial differential control problems, Tikhonov regularization amounts to adding suitably weighted least squares terms of the control variable, or derivatives thereof, to the Lagrangian determining the optimality system. In this note we show that the stabilization methods for discretely ill-posed problems developed in the setting of convection-dominated convection–diffusion problems, can be highly suitable for stabilizing optimal control problems, and that Tikhonov regularization will lead to less accurate discrete solutions. We consider some inverse problems for Poisson’s equation as an illustration and derive new error estimates both for the reconstruction of the solution from the measured data and reconstruction of the source term from the measured data. These estimates include both the effect of the discretization error and error in the measurements.

Keywords: optimal control problem, data assimilation, source identification, finite elements, regularization

(Some figures may appear in colour only in the online journal)

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1. Introduction

In this note we propose an alternative to the classical Tikhonov regularization approach in the finite element approximations of optimal control problems governed by elliptic partial differential equations. We shall, following [5], consider problems of the type

$$J(u, q) \rightarrow \min!, \quad A(u) = f + B(q), \quad (1)$$

where J is a cost functional, A is an elliptic differential operator for the state variable u , and B is an impact operator for the control variable q . Introducing the costate variable λ , this problem can be formulated as finding saddle points for the Lagrangian functional

$$\mathcal{L}(u, q, \lambda) := J(u, q) + (\lambda, A(u) - f - B(q)), \quad (2)$$

where (\cdot, \cdot) denotes the L_2 inner product, determined by the system

$$\begin{cases} \frac{d}{d\epsilon_1} \mathcal{L}(u + \epsilon_1 v, q, \lambda)|_{\epsilon_1=0} = 0, \\ \frac{d}{d\epsilon_2} \mathcal{L}(u, q + \epsilon_2 r, \lambda)|_{\epsilon_2=0} = 0, \\ \frac{d}{d\epsilon_3} \mathcal{L}(u, q, \lambda + \epsilon_3 \mu)|_{\epsilon_3=0} = 0. \end{cases} \quad (3)$$

In a finite element setting, the continuous states, controls and costates $(u, q, \lambda) \in V \times Q \times V$ are replaced by their discrete counterparts $(u^h, q^h, \lambda^h) \in V^h \times Q^h \times V^h$, where V^h and Q^h are finite dimensional counterparts of the appropriate Hilbert spaces V and Q , respectively.

Typically, the cost functional measures some distance between the discrete state and a known or sampled function u_0 over a subdomain $\mathcal{M} \subseteq \Omega$, where $\Omega \subset \mathbb{R}^d$, and $d = 2, 3$ is the polyhedral (polygonal) domain of computation.

$$J(u, q) := \frac{1}{2} \|u - u_0\|_{\mathcal{M}}^2 \quad (4)$$

which may not lead to a well-posed problem. A classical regularization method due to Tikhonov, see [30], is to add a stabilizing functional $n(q, q)$,

$$J(u, q) := \frac{1}{2} \|u - u_0\|_{\mathcal{M}}^2 + n(q, q), \quad (5)$$

where typically

$$n(q, q) := \alpha_0 \|q - q_b\|^2 + \alpha_1 \|\nabla(q - q_b)\|^2, \quad (6)$$

where α_0 and α_1 are regularization parameters and q_b is the background state, or first guess state. The role of the background state is to diminish the inconsistent character of the Tikhonov regularization and implies additional *a priori* knowledge of the system beyond the samples u_0 . In this note we will assume that no such additional *a priori* data is at hand, $q_b = 0$, and that there is no physical justification for the addition of the term $n(q, q)$ —or that the parameters α_0 and α_1 given by the application are too small to provide sufficient stabilization of the system for computational purposes. The actual choice of the parameters α_0, α_1 typically depends on the level of noise in the data—see for example [32, 36]—and can be determined iteratively. Work has also been done using the Morozov discrepancy principle to choose both regularization parameters and discretization space [1, 3, 28]. More generally, in the computation the mesh size is assumed to be small enough for the regularization to stabilize the numerical scheme as well. The rule of thumb is then that the discretization error should be of the order

of the noise level [40, 43]. None of the above references consider analysis of the discretization error and perturbations in a unified setting.

It is well known that an inverse problem based on a linear operator with finite dimensional range is always well posed in the sense of Nashed [39], since the discrete forward operator has a closed range. If the linear system comes from the finite element discretization of an ill-posed PDE, then even if the system happens to be square with non-zero eigenvalues, it will typically be increasingly ill-conditioned as the dimension of the discretization space increases. This ill-conditioning is expressed as non-uniformity in the norms determining the stability properties of the system: as the dimension increases, the stability bounds degenerate. This in its turn may have a detrimental effect on the accuracy of a numerical approximation. Indeed, for a computational PDE this phenomenon is not restricted to ill-posed problems, but arises typically in situations where coercivity arguments fail, see for instance [21, section 5]. The aim of the present paper is to consider the discretizations of PDE problems that are ill-posed in a finite element framework, and provide regularization methods that allow for an analysis that considers both the discretization error and perturbation errors in a unified framework. More precisely, we are interested in problems that have some conditional stability property (see for instance [33, definition 4.3]). The key point is that the existence of a solution implies uniqueness and that a conditional stability estimate is available. The approach that we will follow is to eliminate the Tikhonov regularization on the continuous level and instead regularize the discrete formulation, hence making the regularization part of the computational method in the form of a weakly consistent stabilization. The regularization/stabilization parameter will then be linked to the mesh size and perturbations in the data through the error estimates. The terminology stabilization versus regularization is slightly ambiguous, but in classical numerical analysis the method of modified equations [27] provides a link between these concepts. The accuracy that is achievable depends on the regularity of the exact solution and the size of the perturbations in the data, but is optimal with respect to the order of the method and the stability of the problem. Note in particular that in this case the optimal parameter is independent of the stability of the quantity of interest. To make the comparison with the standard regularization technique easier we also add the analysis outline of such methods, relating the regularization parameter to the mesh-size. In contrast with our proposed method, the optimal choice of parameter for the Tikhonov regularization requires detailed knowledge of the stability properties of the quantity of interest of the continuous problem. Even if this parameter is optimized with respect to the stability of the target quantity, the theoretical convergence order of the finite element method is always superior to that of Tikhonov regularization. On the other hand stabilizations such as those proposed herein change the matrix structure of the Galerkin method by increasing the bandwidth. This raises questions on how to solve the linear systems efficiently, which are more challenging compared to classical Tikhonov regularization.

Finally, let us point out that the use of conditional stability estimates has already been used for the choice of regularizing parameters [16, 34, 35], adaptive approaches for the choice of the parameter were explored in [26, 32], and iterative methods to find an optimal parameter proposed in [32]. The idea of using conditional stability estimates for analysis of the reconstruction method was used, for instance, in [17] for reconstructions using the Landweber iteration and in [29] in the context of variational regularization.

1.1. Model problems

We will discuss two model problems below, both based on a Poisson-type elliptic problem. In the first, data is given in a subset of the bulk instead of on the boundary, and in the second the

solution is known in the domain and we wish to reconstruct the source term. The first problem can be associated with an ill-posed boundary control problem or a data assimilation problem, and the second is related to an ill-posed distributed control problem.

1.1.1. Model problem one: unique continuation. Given a subset $\mathcal{M} \subset \Omega$ that is open, non-empty and distinct, the data $u_0 \in H^1(\mathcal{M})$ and source term $f \in L^2(\Omega)$, we wish to find $u \in H^1(\Omega)$ such that

$$u = u_0 \text{ in } \mathcal{M} \quad (7)$$

and u is a weak solution to

$$-\Delta u = f \text{ in } \Omega. \quad (8)$$

This is not possible for all u_0 and f , but we will assume that the data is such that a solution exists. We will refer to this problem as the *unique continuation problem*. The ill-posedness of this problem follows by the following argument. Consider the operator

$$T : H^1(\Omega) \rightarrow H^{-1}(\Omega) \times H^1(\mathcal{M}), \quad Tu = (\Delta u, u|_{\mathcal{M}}).$$

The best approximate solution to (7) and (8) is given by the pseudoinverse $T^\dagger(f, u_0)$ whenever this is well defined. It is well-known, see [20, proposition 2.4], that T^\dagger is continuous if and only if the range $R(T)$ is closed. In this case T^\dagger is also well-defined on the whole $H^{-1}(\Omega) \times H^1(\mathcal{M})$. It makes sense to say that the minimization problem

$$\min \|Tu - (f, u_0)\|_{H^{-1}(\Omega) \times H^1(\mathcal{M})}$$

is well posed if and only if T^\dagger is continuous.

However, $R(T)$ is not closed. Indeed, to get a contradiction suppose that this is the case. We know that T is injective. Then $\tilde{T} : H^1(\Omega) \rightarrow R(T)$, $\tilde{T} = T$ is bijective. As $R(T)$ is closed, it is a Hilbert space, and $\tilde{T}^{-1} : R(T) \rightarrow H^1(\Omega)$ is continuous by the open mapping theorem. But this implies that

$$\|u\|_{H^1(\Omega)} = \|\tilde{T}^{-1}Tu\|_{H^1(\Omega)} \leq C\|Tu\|_{H^{-1}(\Omega) \times H^1(\mathcal{M})} = C(\|\Delta u\|_{H^{-1}(\Omega)} + \|u\|_{H^1(\mathcal{M})}).$$

Finally by choosing u as the solution of the classical Hadamard counterexample for the Cauchy problem we obtain a contradiction. We can for example take $u_n = \sin(nx) \sinh(ny)$, so that $\Delta u_n = 0$ and assume that \mathcal{M} is centered around the origin; then C cannot be fixed independently of n .

It is known that in case the solution exists, a conditional stability estimate holds. The estimate can be quantified in a three-sphere inequality. Let \mathcal{M} include a ball $B_{r_1}(\mathbf{x}_0)$ with radius r_1 , centered at \mathbf{x}_0 . We can then show the stability of the solution u in $B_{r_2}(\mathbf{x}_0)$, with $r_2 > r_1$ under the *a priori* assumption that $B_{r_2}(\mathbf{x}_0) \subset B_{r_3}(\mathbf{x}_0) \subset \Omega$ and $u \in H^1(\Omega)$ is a weak solution to (8).

Lemma 1 (Three-sphere inequality). Assume that $u : \Omega \rightarrow \mathbb{R}$ is a weak solution of (8) with $f \in H^{-1}(\Omega)$ such that $\|f\|_{H^{-1}(\Omega)} \leq \varepsilon$ for some $\varepsilon > 0$. For every r_1, r_2, r_3, \bar{r} such that $0 < r_1 < r_2 < r_3 < \bar{r}$ and for every $\mathbf{x}_0 \in \Omega$ such that $\text{dist}(\mathbf{x}_0, \partial\Omega) > \bar{r}$ there holds

$$\|u\|_{L^2(B_{r_2})} \leq C \left(\|u\|_{L^2(B_{r_1})} + \varepsilon \right)^\tau \cdot \left(\|u\|_{L^2(B_{r_3})} + \varepsilon \right)^{(1-\tau)} \quad (9)$$

where B_{r_i} , $i = 1, 2, 3$ are balls centered at \mathbf{x}_0 , $C > 0$ and τ , $0 < \tau < 1$ only depend on the geometry of Ω , and the ratios r_2/r_1 and r_3/r_2 .

Proof. For a proof in the non-homogeneous case see Alessandrini *et al* [2, theorem 1.10]. \square

Note that similar results hold in the H^1 -semi-norm—see for example [7]—and the results below can be extended to this case, under suitable modifications of the scheme. The sensitivity on the perturbations in the data is, however, expected to increase, since in this case the H^1 -semi-norm of the perturbations will come into play in the estimates.

We now cast the problem in the form of a constrained, regularized minimization problem: find $u \in H^1(\Omega)$ minimizing

$$\frac{1}{2} \|u - u_0\|_{L_2(\mathcal{M})}^2 + \frac{\alpha}{2} \|\nabla u\|_{L_2(\Omega)}^2 \quad (10)$$

subject to

$$-\Delta u = f, \text{ in } \Omega. \quad (11)$$

Introducing the Lagrange multiplier $\lambda \in H_0^1(\Omega)$, we have the optimality system (see [30, 31]): find $(u_\alpha, \lambda_\alpha) \in H^1(\Omega) \times H_0^1(\Omega)$ such that

$$\int_{\mathcal{M}} u_\alpha v \, d\Omega + \int_{\Omega} \nabla \lambda_\alpha \cdot \nabla v \, d\Omega + \alpha \int_{\Omega} \nabla u_\alpha \cdot \nabla v \, d\Omega = \int_{\mathcal{M}} u_0 v \, d\Omega \quad \forall v \in H^1(\Omega), \quad (12)$$

$$\int_{\Omega} \nabla u_\alpha \cdot \nabla \mu \, d\Omega = \int_{\Omega} f \mu \, d\Omega \quad \forall \mu \in H_0^1(\Omega). \quad (13)$$

For $\alpha > 0$ this system is well-posed by the Babuska–Lax–Milgram lemma [4], but even if u_0 is such that a solution to (11) exists with $u = u_0$ in \mathcal{M} , the solution will in general not be a solution to the optimality system (12) and (13). This is only satisfied by $u = \lim_{\alpha \rightarrow 0} u_\alpha$. Here u denotes the function satisfying (11) and such that $u|_{\mathcal{M}} = u_0$. The regularized solution however must fit the data to order $O(\alpha^{\frac{1}{2}})$. This follows by testing equation (12) with $u_\alpha - u$ and using that $u|_{\mathcal{M}} = u_0$ leading to

$$2\|u_\alpha - u_0\|_{L_2(\mathcal{M})}^2 + \alpha \|\nabla u_\alpha\|_{L_2(\Omega)}^2 + \alpha \|\nabla(u_\alpha - u)\|_{L_2(\Omega)}^2 = \alpha \|\nabla u\|_{L_2(\Omega)}^2.$$

By (13) there holds $\Delta u_\alpha = \Delta u$ independent of α so that $\|\Delta u_\alpha\|_{L^2(\Omega)} = \|\Delta u\|_{L^2(\Omega)}$. Collecting the above bounds we deduce that for all $\alpha > 0$

$$\|u_\alpha - u_0\|_{L_2(\mathcal{M})} + \alpha^{\frac{1}{2}} \|\nabla u_\alpha\|_{L^2(\Omega)} \leq 2\alpha^{\frac{1}{2}} \|\nabla u\|_{L^2(\Omega)} \quad \text{and} \quad \|\Delta u_\alpha\|_{L^2(\Omega)} = \|\Delta u\|_{L^2(\Omega)}. \quad (14)$$

It also follows from this relation that $\|\nabla u_\alpha\|_{L^2(\Omega)}$ is bounded in the limit. Observe that in the special case when $u \in H_0^1(\Omega)$ satisfies (11) with $u|_{\mathcal{M}} = u_0$, then $u_\alpha = u$ for all $\alpha > 0$. This is easily verified by choosing $u_\alpha = u$ and $\lambda_\alpha = -\alpha u$ in (12) and (13).

In the case where $\alpha = 0$ we have the optimality system: find $(u, \lambda) \in H^1(\Omega) \times H_0^1(\Omega)$ such that

$$\int_{\mathcal{M}} u v \, d\Omega + \int_{\Omega} \nabla \lambda \cdot \nabla v \, d\Omega = \int_{\mathcal{M}} u_0 v \, d\Omega \quad \forall v \in H^1(\Omega), \quad (15)$$

$$\int_{\Omega} \nabla u \cdot \nabla \mu \, d\Omega = \int_{\Omega} f \mu \, d\Omega \quad \forall \mu \in H_0^1(\Omega). \quad (16)$$

Clearly the solution to (7) and (8) solves (15) and (16) with $\lambda = 0$. But the unregularized minimization problem is ill-posed, similar to (7) and (8).

Below we will assume that $u_0 \in H^1(\mathcal{M})$ is the unperturbed measurement for which the unique solution exists and consider a numerical method for the approximation of u given some perturbed data, $\tilde{u}_0 := u|_{\mathcal{M}} + \delta u$, with $\delta u \in L^2(\mathcal{M})$; hence $\tilde{u}_0 \in L^2(\mathcal{M})$, which is why we minimize the L^2 -norm, $\|u - u_0\|_{L^2(\mathcal{M})}$.

Remark 1 (Relation to boundary control problems). We can also consider the problem of finding a function $q : \partial\Omega \rightarrow \mathbb{R}$ minimizing

$$\frac{1}{2} \|u(q) - u_0\|_{L^2(\mathcal{M})}^2 + \frac{\alpha}{2} \|\nabla u(q)\|_{L^2(\Omega)}^2 \quad (17)$$

subject to

$$\begin{aligned} -\Delta u &= f, \text{ in } \Omega, \\ u &= q \text{ on } \partial\Omega. \end{aligned} \quad (18)$$

Clearly if q is found, which minimizes (17), the above discussed unique continuation problem is also solved and vice versa. Indeed, by solving the unique continuation problem, we find the optimal q by taking the trace of u . In the boundary control case it may be convenient to make the regularizing least squares term in (17) act directly on q . Finite element methods for this problem have been proposed in [15, 18, 24, 25, 38, 44], for instance, typically with a regularizing term on the form $\|q\|_{L^2(\partial\Omega)}$, but there appears to be no work in the case of the vanishing regularization, using the conditional stability of the limit problem, which is the topic of the present paper.

1.2. Model problem two: source reconstruction

Our second example considers the case where the data is available in the whole domain, $\mathcal{M} \equiv \Omega$, but the source term is unknown and must be reconstructed. The challenge here is that the application of the Laplacian is unstable. This case will be referred to as *source reconstruction* below, but is also related to a distributed control problem. We consider the elementary problem: minimize

$$\frac{1}{2} \|u - u_0\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|q\|_{L^2(\Omega)}^2 \quad (19)$$

subject to

$$-\Delta u = q \text{ in } \Omega, u = 0 \text{ on } \partial\Omega. \quad (20)$$

Here, u_0 is known data and Ω is a convex polygonal (polyhedral) subset of \mathbb{R}^d , $d = 2, 3$, with outward-pointing normal n . We assume that we wish to solve (19) and (20) in the situation where u_0 is a measurement on a system that is of the form (20). This means that if no perturbations are present in the data and measurements are available in every point of Ω , the minimizer for $\alpha = 0$ is $u = u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and an associated $q = -\Delta u \in L^2(\Omega)$ exists so that (20) is satisfied. Also assume that the regularity bound

$$\|u\|_{H^2(\Omega)} \leq C \|q\|_{L^2(\Omega)} \quad (21)$$

holds. Below we will consider the problem of reconstructing q from u_0 , using a stabilized finite element method, and provide an analysis accounting for both the discretization error and the error due to errors in the measured data u_0 .

Introducing the Lagrange multiplier $\lambda \in H_0^1(\Omega)$, we have the optimality system (see [30, 31]): find $(u, q, \lambda) \in H_0^1(\Omega) \times L_2(\Omega) \times H_0^1(\Omega)$ such that

$$\int_{\Omega} u v \, d\Omega + \int_{\Omega} \nabla \lambda \cdot \nabla v \, d\Omega = \int_{\Omega} u_0 v \, d\Omega \quad \forall v \in H_0^1(\Omega), \quad (22)$$

$$\alpha \int_{\Omega} q r \, d\Omega + \int_{\Omega} \lambda r \, d\Omega = 0 \quad \forall r \in L_2(\Omega), \quad (23)$$

$$\int_{\Omega} \nabla u \cdot \nabla \mu \, d\Omega = \int_{\Omega} q \mu \, d\Omega \quad \forall \mu \in H_0^1(\Omega). \quad (24)$$

We note here that the trace of λ is zero on the boundary and that this introduces an artificial boundary condition on q through the regularization term. Since q is equal to λ almost everywhere we also observe that the regularization in L^2 actually imposes an H^1 -regularity on the source term. These artefacts carry over to the approximate solution when the regularized system is discretized. Once again we are interested here in the optimality system of (19) and (20) when $\alpha = 0$: find $(u, q, \lambda) \in H_0^1(\Omega) \times L_2(\Omega) \times H_0^1(\Omega)$ such that

$$\int_{\Omega} u v \, d\Omega + \int_{\Omega} \nabla \lambda \cdot \nabla v \, d\Omega = \int_{\Omega} u_0 v \, d\Omega \quad \forall v \in H_0^1(\Omega), \quad (25)$$

$$\int_{\Omega} \lambda r \, d\Omega = 0 \quad \forall r \in L_2(\Omega), \quad (26)$$

$$\int_{\Omega} \nabla u \cdot \nabla \mu \, d\Omega = \int_{\Omega} q \mu \, d\Omega \quad \forall \mu \in H_0^1(\Omega). \quad (27)$$

Here we also assume that only the perturbed data $\tilde{u}_0 = u|_{\mathcal{M}} + \delta u$ with $\delta u \in L^2(\mathcal{M})$ is available for the reconstruction, and we will estimate quantitatively the effect of δu on the bounds below.

Remark 2 (Relation to distributed control problems). The related distributed control problem is that of finding a function $q : \Omega \rightarrow \mathbb{R}$ minimizing

$$\frac{1}{2} \|u(q) - u_0\|_{L_2(\mathcal{M})}^2 + \frac{\alpha}{2} \|q\|_{L_2(\Omega)}^2 \quad (28)$$

subject to

$$\begin{aligned} -\Delta u &= q, \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \quad (29)$$

Here we consider the simplest context where the data u_0 is known over the whole domain Ω . Finite element methods for the distributed control problems have been considered in [6, 22, 23, 37], for instance. Often some additional pointwise constraints on the control variable q are introduced. To the best of our knowledge no works consider the situation of vanishing regularization, which is at the interface between optimal control and inverse problems.

2. Discretization of ill-posed problems

Since the equations to which we wish to compute solutions are ill-posed, both the well-posedness of the forward and the adjoint problems may be compromised, and a naive discretization of the problem can not be expected to be successful. Indeed it is known, for instance, that the finite element equivalent of the unique extension of harmonic functions, which is at the basis of the stability of lemma 1, does not hold. To rectify the situation we draw on the experiences of the stabilization of well-posed, but numerically unstable problems, such as indefinite problems, convection–diffusion equations or Stokes equations [8, 12, 13] and propose to stabilize the discrete ill-posed problems by penalizing the fluctuations of the solution. This results in a regularization that has no obvious interpretation on the continuous level, but ensures that the discrete system is invertible. The stability of the continuous problem may then be applied to the error of the computational approximation, resulting in error estimates.

2.1. Derivation of the discrete model

Let $\{\mathcal{T}_h\}_h$ denote a family of shape regular and quasi-uniform tessellations of Ω in non-overlapping simplices, such that for any two different simplices $K, K' \in \mathcal{T}_h$, $K \cap K'$ consists of either the empty set, a common face/edge or a common vertex. The outwardly pointing normal of a simplex K will be denoted n_K and the diameter h_K . The global mesh parameter of \mathcal{T}_h is defined by $h := \max_{K \in \mathcal{T}_h} h_K$. We denote the set of interior element faces F in \mathcal{T}_h by \mathcal{F}_I . To each face we associate its diameter h_F and a normal n_F , whose orientation is arbitrary but fixed. We define the standard finite element space of continuous piecewise affine functions on \mathcal{T}_h

$$V_h := \{v_h \in C^0(\bar{\Omega}) : v_h|_K \in P_1(K), \forall K \in \mathcal{T}_h\},$$

where $P_1(K)$ denotes the set of affine functions on K . We also define $V_h^0 := V_h \cap H_0^1(\Omega)$. The following bilinear forms defined on $V_h \times V_h$ will be useful for the formulation of our finite element methods,

$$m_X(v_h, w_h) := \int_X v_h w_h \, d\Omega, \text{ for } X \subseteq \Omega, \quad (30)$$

$$s_i(v_h, w_h) := \sum_{F \in \mathcal{F}_I} \gamma_i \int_F h_F^i [\nabla v_h \cdot n_F] [\nabla w_h \cdot n_F] \, ds \quad (31)$$

where $[[y_h]]_F := \lim_{\epsilon \rightarrow 0^+} (y_h(x - \epsilon n_F) - y_h(x + \epsilon n_F))$ denotes the jump of the quantity y_h over the face F , with the normal n_F , and $\gamma_i \in \mathbb{R}^+$ denotes a dimensionless parameter independent of h . Finally

$$a(v_h, w_h) := \int_{\Omega} \nabla v_h \cdot \nabla w_h \, d\Omega. \quad (32)$$

2.2. Discretization of the unique continuation problem

If we write the minimization problem (10) and (11) on the discrete space V_h , we may replace the term $\|\nabla u\|_{L^2(\Omega)}^2$ with the form $s_1(u_h, u_h)$ defined in (31). We then want to find $u_h \in V_h$ minimizing

$$\frac{1}{2} \|u_h - u_0\|_{L^2(\mathcal{M})}^2 + \frac{1}{2} s_1(u_h, u_h) \quad (33)$$

under the constraint

$$a(u_h, v_h) = m_\Omega(f, v_h), \quad \forall v_h \in V_h^0.$$

Observe that the constraint equation is underdetermined since the trial space is larger than the test space.

Remark 3. In the discrete setting we may take both the trial and the test space to be V_h , but in this case, for consistency, we must add a boundary term to the definition (32) of a leading to the modified form

$$a(v_h, w_h) := \int_{\Omega} \nabla v_h \cdot \nabla w_h \, d\Omega - \int_{\partial\Omega} \nabla v_h \cdot n w_h \, ds.$$

This term does not make sense on the continuous level, but can be included in the analysis below after minor modifications. For conciseness we leave the details to the reader.

Considering the optimality of the above discrete minimization problem we obtain the finite element formulation: find $u_h, \lambda_h \in V_h \times V_h^0$ such that

$$m_{\mathcal{M}}(u_h, v_h) + s_1(u_h, v_h) + a(v_h, \lambda_h) = m_{\mathcal{M}}(\tilde{u}_0, v_h) \quad \forall v_h \in V_h, \quad (34)$$

$$a(u_h, \mu_h) = m_\Omega(\tilde{f}, \mu_h) \quad \forall \mu_h \in V_h^0, \quad (35)$$

where $\tilde{f} := f + \delta f$ and $\tilde{u}_0 := u_0 + \delta u_0$, with $\delta f \in H^{-1}(\Omega)$ and $\delta u_0 \in L^2(\mathcal{M})$ denoting measurement errors in the source term and data.

This may then be written on the compact form, find $u_h, \lambda_h \in \mathcal{V}_h^{UC}$, with $\mathcal{V}_h^{UC} := V_h \times V_h^0$ such that

$$A_{UC}[(u_h, \lambda_h), (v_h, \mu_h)] = m_\Omega(\tilde{f}, \mu_h) + m_{\mathcal{M}}(\tilde{u}_0, v_h), \quad \forall v_h, \mu_h \in \mathcal{V}_h^{UC},$$

with

$$A_{UC}[(u_h, \lambda_h), (v_h, \mu_h)] := m_{\mathcal{M}}(u_h, v_h) + s_1(u_h, v_h) + a(v_h, \lambda_h) + a(u_h, \mu_h).$$

Remark 4. An analogy can be made with the stabilized finite element discretizations for singularly perturbed convection–diffusion equations of the form

$$u + \mathbf{a} \cdot \nabla u - \varepsilon \Delta u = f, \quad (36)$$

where $\mathbf{a} \in [W^{1,\infty}(\Omega)]^d$ and $\varepsilon \in \mathbb{R}^+$. When the Péclet number ($\text{Pe}(h) = (|\mathbf{a}|h)/\varepsilon$, where h is the mesh size), is large, the H^1 -stability of the scheme $\|\varepsilon^{\frac{1}{2}} \nabla u_h\|_{\Omega} \leq \varepsilon^{-\frac{1}{2}} \|f\|_{L^2(\Omega)}$ obtained by the standard energy estimate is insufficient to ensure the stability of the discretization and spurious oscillations appear that compromise the accuracy. Reference [12] proposed adding a stabilizing term on the form $s_2(\cdot, \cdot)$ to the standard FEM formulation, which was shown to improve the estimates in the spirit of other known methods for this problem (see [41]). The key to the enhanced stability is that the addition of the stabilizing term makes the following stability estimate hold

$$\|u_h\|_{L^2(\Omega)} + \|h^{\frac{1}{2}} \mathbf{a} \cdot \nabla u_h\|_{L^2(\Omega)} + \|\varepsilon^{\frac{1}{2}} \nabla u_h\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}.$$

The stabilization improves the robustness in the limit $\varepsilon \rightarrow 0$, and in the limit $h \rightarrow 0$ it limits the rate of blow-up of $\|\mathbf{a} \cdot \nabla u_h\|_{L^2(\Omega)}$.

If we compare it with the unique continuation problem, the unsymmetric operator is the (discrete) Laplacian $\Delta_h u_h \in V_h^0$ defined by

$$m_\Omega(\Delta_h u_h, v_h) = a(u_h, v_h) \quad \forall v_h \in V_h^0.$$

Taking $v_h = h^2 \Delta_h u_h$ we obtain

$$\|h \Delta_h u_h\|_{L^2(\Omega)}^2 = m_\Omega(\Delta_h u_h, h^2 \Delta_h u_h) = a(u_h, h^2 \Delta_h u_h).$$

One may then easily show, using integration by parts and the techniques introduced below (anticipating the inequalities (41)), that

$$a(u_h, h^2 \Delta_h u_h) \leq \sum_{F \in \mathcal{F}_I} \int_F h^{\frac{1}{2}} \|\nabla u_h \cdot n_F\| h^{-\frac{1}{2}} |h^2 \Delta_h u_h| \leq C |u_h|_{s_1} \|h \Delta_h u_h\|_{L^2(\Omega)}.$$

It follows that

$$\|h \Delta_h u_h\|_{L^2(\Omega)} \leq C |u_h|_{s_1},$$

showing that the addition of the penalty to the gradient fluctuation enhances the stability on the discrete level, similar to the case of convection–diffusion equations. This estimate, however, does not make sense for continuous equations.

2.3. Discretization of the source reconstruction problem

If we write the minimization problem (19) and (20) on the discrete space V_h , we may replace the term $\|q_h\|_{L^2(\Omega)}^2$ by the form $s_5(q_h, q_h)$ defined in (31). We then want to find $u_h \in V_h^0$ minimizing

$$\frac{1}{2} \|u_h - u_0\|_{L^2(\Omega)}^2 + \frac{1}{2} s_1(u_h, u_h) + \frac{1}{2} s_5(q_h, q_h) \quad (37)$$

under the constraint

$$a(u_h, v_h) = m_\Omega(q_h, v_h), \quad \forall v_h \in V_h^0.$$

Observe that we also need to stabilize the solution u_h . This is due to the large kernel of the operator s_5 , which makes the natural stability of the constraint equation insufficient. Below we will also discuss other more conventional regularizations in our framework. Considering the optimality of the above discrete minimization problem we obtain the finite element formulation: find $u_h, q_h, \lambda_h \in V_h^0 \times V_h \times V_h^0$ such that

$$m_\Omega(u_h, v_h) + s_1(u_h, v_h) + a(v_h, \lambda_h) = m_\Omega(\tilde{u}_0, v_h) \quad \forall v_h \in V_h^0, \quad (38)$$

where we have used the perturbed data $\tilde{u}_0 = u_0 + \delta u$, where $\delta u \in L^2(\Omega)$,

$$m_\Omega(\lambda_h, w_h) - s_5(q_h, w_h) = 0 \quad \forall w_h \in V_h \quad (39)$$

and

$$\bar{a}(u_h, \mu_h) = m_\Omega(q_h, \mu_h) \quad \forall \mu_h \in V_h^0 \quad (40)$$

with bilinear forms given by (30)–(32) above. Below we will distinguish the stabilization parameters of $s_1(\cdot, \cdot)$ and $s_5(\cdot, \cdot)$ and denote them by γ_1 and γ_5 respectively.

This may then be written on the compact form: find $u_h, q_h, \lambda_h \in \mathcal{V}_h^{SR}$ with $\mathcal{V}_h^{SR} := V_h^0 \times V_h \times V_h^0$, such that

$$A_{SR}[(u_h, q_h, \lambda_h), (v_h, w_h, \mu_h)] = m_\Omega(\tilde{u}_0, v_h), \quad \forall v_h, r_h, \mu_h \in \mathcal{V}_h^{SR},$$

with

$$\begin{aligned} A_{SR}[(u_h, q_h, \lambda_h), (v_h, w_h, \mu_h)] := & m_\Omega(u_h, v_h) + s_1(u_h, v_h) \\ & + a(v_h, \lambda_h) + s_5(q_h, w_h) - m_\Omega(\lambda_h, w_h) \\ & - a(u_h, \mu_h) + m_\Omega(q_h, \mu_h). \end{aligned}$$

3. Preliminary technical results

First we define the semi-norms associated with the stabilization operator $s_i(\cdot, \cdot)$,

$$|x_h|_{s_i} = s_i(x_h, x_h)^{\frac{1}{2}}, \quad \forall x_h \in V_h.$$

We recall the following well-known inverse and trace inequalities (see for instance [19, section 1.4.3])

$$\begin{aligned} \|v\|_{L^2(\partial K)} &\leq C_t(h^{-\frac{1}{2}}\|v\|_{L^2(K)} + h^{\frac{1}{2}}\|\nabla v\|_{L^2(K)}), \quad \forall v \in H^1(K), \\ h_K^{\frac{1}{2}}\|\nabla v_h \cdot n_K\|_{L^2(\partial K)} &\leq C_\iota\|\nabla v_h\|_{L^2(K)}, \quad \forall v_h \in P_1(K), \\ h_K\|\nabla v_h\|_{L^2(K)} + h_K^{\frac{1}{2}}\|v_h\|_{L^2(\partial K)} &\leq C_\iota\|v_h\|_{L^2(K)}, \quad \forall v_h \in P_1(K). \end{aligned} \quad (41)$$

As an immediate consequence of (41) we have the following stabilities for some $C_{si} > 0$ depending only on the mesh geometry:

$$|x_h|_{s_i} \leq C_{si}h^{\frac{i-3}{2}}\|x_h\|_{L^2(\Omega)}, \quad |x_h|_{s_i} \leq C_{si}h^{\frac{i-1}{2}}\|\nabla x_h\|_{L^2(\Omega)}. \quad (42)$$

This follows from the definition of $s_i(\cdot, \cdot)$ and the inverse inequalities (41)

$$|x_h|_{s_i}^2 := \sum_{F \in \mathcal{F}_i} \|h_F^{i/2} [\nabla x_h \cdot n_F]\|_{L^2(F)}^2 \leq C \sum_{K \in \mathcal{T}_h} h_K^{i-1} \|\nabla x_h\|_{L^2(K)}^2 \leq C \sum_{K \in \mathcal{T}_h} h_K^{i-3} \|x_h\|_{L^2(K)}^2.$$

Let $i_h : H^2(\Omega) \rightarrow V_h$ denote the Scott–Zhang interpolant and $\pi_h : L^2 \rightarrow V_h$ and $\pi_h^0 : L^2 \rightarrow V_h^0$ denote the L^2 -projections on the respective finite element spaces. The following error estimate is known to hold both for i_h and π_h :

$$\|u - i_h u\|_{L^2(\Omega)} + h\|\nabla(u - i_h u)\|_{L^2(\Omega)} \leq Ch^t|u|_{H^t(\Omega)}, \quad t = 1, 2. \quad (43)$$

To prove the stability of our formulations below we need to show that for any function $v_h \in V_h$ the L^2 -norm is equivalent to $\|\pi_h^0 v_h\|_{L^2(\Omega)} + |v_h|_{s_3}$. We prove the result in this technical lemma.

Lemma 2. *There exists $C_p > 0$ such that for all $v_h \in V_h$ there holds*

$$h\|v_h\|_{H^1(\Omega)} \leq C_p(\|v_h\|_{L^2(\mathcal{M})} + |v_h|_{s_1}). \quad (44)$$

There exists $c_1, c_2 > 0$ such that for any function $v_h \in V_h$

$$c_1(\|\pi_h^0 v_h\|_{L^2(\Omega)} + |v_h|_{s_3}) \leq \|v_h\|_{L^2(\Omega)} \leq c_2(\|\pi_h^0 v_h\|_{L^2(\Omega)} + |v_h|_{s_3}). \quad (45)$$

Proof. The discrete Poincaré type inequality (44) may be proved using a compactness argument similar to that of [14]. To lessen the technical detail, we use here an approach with continuous Poincaré inequalities and discrete interpolation instead. Let $\mathcal{I}_h : \nabla V_h \mapsto [V_h]^d$ be a quasi-interpolation operator [11, section 5] such that

$$\|\nabla v_h - \mathcal{I}_h \nabla v_h\|_{L^2(\Omega)} \leq C|v_h|_{s_1}, \quad \|\mathcal{I}_h \nabla v_h\|_{L^2(K)} \leq C\|\nabla v_h\|_{L^2(\Delta_K)} \quad (46)$$

where $\Delta_K := \cup_{K' : K \cap K' \neq \emptyset} K'$. The following Poincaré inequality is well known (see [21, lemma B.63]). If $f : H^1(\Omega) \mapsto \mathbb{R}$ is a linear functional that is non-zero for constant functions then

$$\|u\|_{H^1(\Omega)} \leq C_P(|f(u)| + \|\nabla u\|_{L^2(\Omega)}), \quad \forall u \in H^1(\Omega).$$

For instance, we may take

$$f(u) = \int_{\mathcal{M}} u \, d\Omega \leq C\|u\|_{L^2(\mathcal{M})}.$$

As an immediate consequence we have the bound

$$\|v_h\|_{H^1(\Omega)} \leq C(\|v_h\|_{L^2(\mathcal{M})} + \|\nabla v_h\|_{L^2(\Omega)}). \quad (47)$$

Now let $\mathcal{M}_{\text{int}} \subset \mathcal{M}$ be the set of interior triangles of \mathcal{M} ,

$$\mathcal{M}_{\text{int}} := \{K \in \mathcal{T}_h : \Delta_K \subset \mathcal{M}\}.$$

It then follows by the stability of \mathcal{I}_h that $\|\mathcal{I}_h \nabla v_h\|_{L^2(\mathcal{M}_{\text{int}})} \leq C\|\nabla v_h\|_{L^2(\mathcal{M})}$. Adding and subtracting $\mathcal{I}_h \nabla v$ in the second term on the right-hand side of (47) gives

$$\begin{aligned} \|v_h\|_{H^1(\Omega)} &\leq C(\|v_h\|_{L^2(\mathcal{M})} + \|\nabla v_h - \mathcal{I}_h \nabla v_h\|_{L^2(\Omega)} + \|\mathcal{I}_h \nabla v_h\|_{L^2(\Omega)}) \\ &\leq C(\|v_h\|_{L^2(\mathcal{M})} + |v_h|_{s_1} + \|\mathcal{I}_h \nabla v_h\|_{L^2(\Omega)}) \end{aligned} \quad (48)$$

where we used (46) in the second inequality. For the third term on the right-hand side of (48) we once again use Poincaré's inequality, the stability of \mathcal{I}_h and the inverse inequality (41) to conclude that

$$\begin{aligned} \|\mathcal{I}_h \nabla v_h\|_{L^2(\Omega)} &\leq C(\|\mathcal{I}_h \nabla v_h\|_{L^2(\mathcal{M}_{\text{int}})} + \|\nabla \mathcal{I}_h \nabla v_h\|_{L^2(\Omega)}) \\ &\leq C(h^{-1}\|v_h\|_{L^2(\mathcal{M})} + \|\nabla \mathcal{I}_h \nabla v_h\|_{L^2(\Omega)}). \end{aligned}$$

Using the fact that ∇v_h is constant on each element we may write

$$\|\nabla \mathcal{I}_h \nabla v_h\|_{L^2(\Omega)}^2 = \sum_{K \in \mathcal{T}_h} \|\nabla(\nabla v_h - \mathcal{I}_h \nabla v_h)\|_{L^2(K)}^2.$$

Then using the inverse inequality (41) of each term in the sum, and the left relation of (46) it follows that

$$\|\nabla \mathcal{I}_h \nabla v_h\|_{L^2(\Omega)}^2 \leq Ch^{-2} \|\nabla v_h - \mathcal{I}_h \nabla v_h\|_{L^2(\Omega)}^2 \leq Ch^{-2} |v_h|_{s_1}^2.$$

Collecting these bounds we have shown that

$$\|\mathcal{I}_h \nabla v_h\|_{L^2(\Omega)} \leq Ch^{-1}(\|v_h\|_{L^2(\mathcal{M})} + |v_h|_{s_1}),$$

which combined with (48) gives (assuming that $h < 1$)

$$h\|v_h\|_{H^1(\Omega)} \leq C(h+1)(\|v_h\|_{L^2(\mathcal{M})} + |v_h|_{s_1}) \leq 2C(\|v_h\|_{L^2(\mathcal{M})} + |v_h|_{s_1})$$

by which we have proven (44).

The lower bound of (45) is immediate by the stability of the L^2 -projection and the inverse and trace inequalities of equation (41). To prove the upper bound we write

$$\|v_h\|_{L^2(\Omega)}^2 = \|\pi_h^0 v_h\|_{L^2(\Omega)}^2 + \|v_h - \pi_h^0 v_h\|_{L^2(\Omega)}^2$$

and let $w_h := v_h - \pi_h^0 v_h$. We will now prove that

$$\|w_h\|_{L^2(\Omega)}^2 \leq |w_h|_{s_3}^2 \quad (49)$$

from which the upper bound follows, since by the triangle inequality followed by the first inequality of (42), with $i = 3$,

$$|w_h|_{s_3} \leq |v_h|_{s_3} + |\pi_h^0 v_h|_{s_3} \leq |v_h|_{s_3} + C_{st} \|\pi_h^0 v_h\|_{L^2(\Omega)}.$$

To prove (49) we first define the support of a nodal basis function φ_i by $\Omega_i := \{x \in \Omega : \varphi_i(x) > 0\}$. Then let \mathcal{N}_I denote the set of indices of basis functions that are in the interior of the domain, that is, for each value $i \in \mathcal{N}_I$ the closure of the support of the associated basis function has an empty intersection with the boundary, $\overline{\Omega_i} \cap \partial\Omega = \emptyset$. For each $k \in \mathcal{N}_I$ we define the macro-patch $\mathcal{P}_k := \cup_{j: \Omega_j \cap \Omega_k \neq \emptyset} \Omega_j$. This means that \mathcal{P}_k consists of Ω_k and any other patch Ω_j sharing two triangles (in 2D) or several tetrahedra (in 3D) with Ω_k . Since \mathcal{T}_h is shape regular we may map the patch \mathcal{P}_k to a shape regular $\tilde{\mathcal{P}}_k$ such that $\text{diam}(\tilde{\mathcal{P}}_k) = 1$. We define the linear map $B : \tilde{\mathcal{P}}_k \mapsto \mathcal{P}_k$ and observe that $\det(B) \sim h^d$ and $\|B\|_F \sim h$, (where $\|\cdot\|_F$ denotes the Frobenius norm). Let $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{N}}$ denote the set of interior faces and interior nodes respectively of $\tilde{\mathcal{P}}_k$ and define the scalar product on $\tilde{\mathcal{P}}_k$

$$(\tilde{v}_h, \tilde{y}_h)_{\tilde{\mathcal{P}}_k} = \int_{\tilde{\mathcal{P}}_k} \tilde{v}_h \tilde{y}_h d\tilde{x}.$$

Clearly, since B is affine, and $\tilde{v}_h = v_h \circ B$, $\tilde{y}_h = y_h \circ B$ then

$$(v_h, y_h)_{\mathcal{P}_k} = \det(B) (\tilde{v}_h, \tilde{y}_h)_{\tilde{\mathcal{P}}_k}.$$

Let $\tilde{\varphi}_j$ denote the mapped basis function $\varphi_j \circ B$ and let \tilde{V}_k denote the space of piecewise affine functions such that for all $\tilde{v}_h \in \tilde{V}_k$ there exists $v_h \in V_h$ such that $\tilde{v}_h = (1 - \pi_0)v_h|_{\mathcal{P}_k} \circ B$. It follows that for all $\tilde{v}_h \in \tilde{V}_k$ there holds

$$(\tilde{v}_h, \tilde{\varphi}_j)_{\tilde{\mathcal{P}}_k} = 0 \quad (50)$$

for all $j \in \tilde{\mathcal{N}}$. Now define the semi-norm $|\cdot|_{s, \tilde{\mathcal{P}}_k}$ by

$$|\tilde{v}_h|_{s, \tilde{\mathcal{P}}_k}^2 := \sum_{\tilde{F} \in \tilde{\mathcal{F}}} \int_{\tilde{F}} [\tilde{\nabla} \tilde{v}_h]^2 d\tilde{s}.$$

We now prove that $|\cdot|_{s, \tilde{\mathcal{P}}_k}$ is a norm on \tilde{V}_k . It is clearly a semi-norm so we only need to prove that $|\tilde{v}_h|_{s, \tilde{\mathcal{P}}_k} = 0$ implies $\tilde{v}_h|_{\tilde{\mathcal{P}}_k} = 0$ for all $\tilde{v}_h \in \tilde{V}_k$. If $|\tilde{v}_h|_{s, \tilde{\mathcal{P}}_k} = 0$ then \tilde{v}_h is an affine function on $\tilde{\mathcal{P}}_k$. It is straightforward to check that the only affine function that can satisfy (50) is the zero

function. To be precise, this is because otherwise \tilde{v}_h has to be odd with respect to the center of mass of all the basis functions—but this is impossible since the mesh is non-degenerate. It follows that the following bound holds:

$$\|\tilde{v}_h\|_{L^2(\tilde{\mathcal{P}}_k)} \leq \tilde{C}|\tilde{v}_h|_{s,\tilde{\mathcal{P}}}, \quad \forall \tilde{v}_h \in \tilde{V}_k.$$

The constant of the estimate depends on the shape regularity. We define

$$|v_h|_{s_3,\mathcal{P}_k}^2 := \sum_{F \in \mathcal{F}_k} \int_F h_F^3 \llbracket \nabla v_h \cdot n_F \rrbracket^2 ds,$$

where \mathcal{F}_k denotes the set of interior faces of \mathcal{P}_k . Observe that since $|\llbracket B^T \nabla v_h \rrbracket| \leq \|B\|_F |\llbracket \nabla v_h \rrbracket|$ we have the bound

$$\begin{aligned} \int_{\tilde{F}} \llbracket \tilde{\nabla} \tilde{v}_h \rrbracket^2 d\tilde{s} &\leq Ch^{1-d} \int_F \llbracket B^T \nabla v_h \rrbracket^2 ds \leq Ch^{3-d} \int_F \llbracket \nabla v_h \rrbracket^2 ds \\ &\leq C \det(B)^{-1} \int_F h_F^3 \llbracket \nabla v_h \cdot n_F \rrbracket^2 ds. \end{aligned}$$

Therefore, by scaling back to the physical geometry, we obtain that there exists $C > 0$ depending only on the local mesh geometry such that

$$\|w_h\|_{L^2(\mathcal{P}_k)} = \det(B)^{\frac{1}{2}} \|\tilde{w}_h\|_{L^2(\tilde{\mathcal{P}}_k)} \leq \det(B)^{\frac{1}{2}} \tilde{C} |\tilde{w}_h|_{s,\tilde{\mathcal{P}}_k} \leq C |w_h|_{s_3,\mathcal{P}_k}.$$

To conclude, we observe that since the overlap between different patches \mathcal{P}_k is bounded uniformly in h there holds

$$\|w_h\|_{L^2(\Omega)}^2 \leq \sum_{k \in \mathcal{N}_I} \|w_h\|_{L^2(\mathcal{P}_k)}^2 \leq C \sum_{k \in \mathcal{N}_I} |w_h|_{s_3,\mathcal{P}_k}^2 \leq C |w_h|_{s_3}^2,$$

which is the desired inequality. \square

4. Error analysis—unique continuation

In this section we will derive error estimates for the formulation (34) and (35) applying techniques from the stabilized FEM together with the continuous stability estimate lemma 1. Assuming that for $\alpha = 0$, the unique continuation problem (7) and (8) admits a solution $u \in H^2(\Omega)$, below we obtain an estimate of the type:

$$\|u - u_h\|_{L^2(B_{r_2})} + \|\nabla(u - u_h)\|_{L^2(B_{r_2})} \leq Ch^\tau, \quad (51)$$

drawing on ideas from [9]. Observe that this estimate reflects the Hölder stability of the continuous problem. To put this in perspective we will discuss briefly the effect of discretizing (10) and (11) when $\alpha > 0$ and approximating the combined regularization and discretization error as α and h go to zero, for previous work, in this spirit we refer to [45]. In this case the solution u_α of the well-posed regularized problem acts as an intermediate function in the analysis. First the stability of the continuous problem is used to assess the effect of regularization, assuming as before the existence of a unique solution when $\alpha = 0$. The application of lemma 1 leads to

$$\|u - u_\alpha\|_{L^2(B_{r_2})} \leq C\alpha^{\frac{\tau}{2}}$$

using linearity and the bound (14). Then the computational error in the finite element approximation of u_α must be approximated. Using standard arguments (see appendix A) assuming $h^2 < \alpha < 1$ we obtain

$$\|\nabla(u_\alpha - u_{\alpha h})\|_\Omega \leq C\alpha^{-\frac{1}{2}}h \quad \text{and} \quad \|u_\alpha - u_{\alpha h}\|_\Omega \leq C\alpha^{-1}h^2. \quad (52)$$

Using a triangle inequality we then obtain

$$\|\nabla(u - u_{\alpha h})\|_{L^2(B_{r_2})} \leq \|\nabla(u - u_\alpha)\|_{L^2(B_{r_2})} + \|\nabla(u_\alpha - u_{\alpha h})\|_{L^2(B_{r_2})} \leq C \left(\alpha^{\frac{\tau}{2}} + \frac{h}{\alpha^{\frac{1}{2}}} \right)$$

and

$$\|u - u_{\alpha h}\|_{L^2(B_{r_2})} \leq \|u - u_\alpha\|_{L^2(B_{r_2})} + \|u_\alpha - u_{\alpha h}\|_{L^2(B_{r_2})} \leq C \left(\alpha^{\frac{\tau}{2}} + \frac{h^2}{\alpha} \right).$$

We then see that the optimal regularization parameter (neglecting the multiplicative constants) is obtained by setting

$$\alpha^{\frac{\tau}{2}} = \frac{h}{\alpha^{1/2}} \rightarrow \alpha = h^{\frac{2}{\tau+1}}$$

in the H^1 -case, with the resulting convergence order

$$\|\nabla(u - u_{\alpha h})\|_{L^2(B_{r_2})} \leq Ch^{\frac{\tau}{\tau+1}}.$$

In the L^2 -case we obtain similarly

$$\alpha = h^{\frac{4}{\tau+2}} \text{ and } \|u - u_{\alpha h}\|_{L^2(B_{r_2})} \leq Ch^{\frac{2\tau}{\tau+2}}.$$

Comparing this with the anticipated result (51) we see that the formal order of the Tikhonov regularized method using the optimal choice of α is a factor $(1 + \tau)^{-1}$ worse than that of (51) for the H^1 -error estimate and a factor $2/(2 + \tau)$ worse in the L^2 -norm. Another important consequence of the above sketch is that the optimal choice of the parameter α in the case of a regularized continuous problem depends on the coefficient τ of the conditional stability that is generally unknown.

For the analysis we introduce the triple norm

$$|||(v_h, \mu_h)|||_{UC} := \|v_h\|_{L^2(\mathcal{M})} + \|hv_h\|_{H^1(\Omega)} + |v_h|_{s_1} + \|\mu_h\|_{H^1(\Omega)} \quad (53)$$

where

$$|x_h|_{s_1} := s_1(x_h, x_h)^{\frac{1}{2}}.$$

Observe that the terms in the above norm do not have matching dimensions. Indeed, there is a constant of the dimension of an inverse length scale present in the first two terms on the right-hand side of (53). In the term over $L^2(\mathcal{M})$ this is to avoid a too strong penalty on the possibly perturbed data, and in the second term on the right-hand side it comes from the application of the discrete Poincaré inequality (44). Stability in the norm (53) is sufficient to deduce the existence of a discrete solution to the system (34) and (35); however, the norm is too weak to be useful for error estimates.

Using (43) and (41) it is straightforward to show the following approximation estimate, for all $v \in H^2(\Omega)$,

$$|||(v - i_h v, 0)|||_{UC} \leq Ch\|v\|_{H^2(\Omega)}. \quad (54)$$

First we note that

$$|||(v - i_h v, 0)|||_{UC} \leq 2\|v - i_h v\|_{H^1(\Omega)} + |v - i_h v|_{s_1}.$$

The bound of the first term on the right-hand side follows using standard interpolation. For the second term we apply the trace inequality of (41) (first inequality), followed by interpolation to obtain

$$\begin{aligned} |v - i_h v|_{s_1}^2 &= \gamma_1 \|h^{\frac{1}{2}} \llbracket \nabla(v - i_h v) \cdot n \rrbracket\|_{\mathcal{F}}^2 \\ &\leq C_I \gamma_1 (\|\nabla(v - i_h v)\|_{L^2(\Omega)} + h|v|_{H^2(\Omega)}) \leq CC_I \gamma_1 h|v|_{H^2(\Omega)}. \end{aligned} \quad (55)$$

The analysis takes the following form, following the framework of [10]. First we prove the inf-sup stability of the form $A_{UC}[\cdot, \cdot]$ in the norm (53). From this the existence of a discrete solution to the linear system follows. Then we show an error estimate in the norm (53) that is independent of the stability of the unique continuation problem and gives convergence rates for the residuals of the approximation. Finally, we show that the error satisfies an equation of the type (8), with the right-hand side given by the residual. The *a priori* error estimates on the residual together with the assumed *a priori* estimate on the exact solution allow us to deduce the error bounds through lemma 1.

Proposition 1. *There exists $c_s > 0$ such that for all $(w_h, \varsigma_h) \in \mathcal{V}_h^{UC}$ there holds*

$$c_s |||(w_h, \varsigma_h)|||_{UC} \leq \sup_{(v_h, \mu_h) \in \mathcal{V}_h^{UC}} \frac{A_{UC}[(w_h, \varsigma_h), (v_h, \mu_h)]}{|||(v_h, \mu_h)|||_{UC}}.$$

Proof. First observe that for $\kappa \in \mathbb{R}^+$ we may write

$$\begin{aligned} A_{UC}[(w_h, \varsigma_h), (w_h + \kappa \varsigma_h, -\varsigma_h)] &= \|w_h\|_{L^2(\mathcal{M})}^2 + |w_h|_{s_1}^2 + \kappa \|\nabla \varsigma_h\|_{L^2(\Omega)}^2 \\ &\quad + \kappa m_{\mathcal{M}}(w_h, \varsigma_h) + \kappa s_1(w_h, \varsigma_h). \end{aligned}$$

Using the Cauchy–Schwarz inequality, an arithmetic-geometric inequality and the inverse inequality (41), in the last term of the right-hand side we get

$$\kappa s_1(w_h, \varsigma_h) \leq \frac{1}{2} |w_h|_{s_1}^2 + \frac{1}{2} \kappa^2 \gamma C_t^2 \|\nabla \varsigma_h\|_{L^2(\Omega)}^2$$

and similarly, using in addition the Poincaré inequality $\|\varsigma_h\|_{L^2(\Omega)} \leq C_p \|\nabla \varsigma_h\|_{L^2(\Omega)}$,

$$\kappa m_{\mathcal{M}}(w_h, \varsigma_h) \leq \frac{1}{2} \|w_h\|_{L^2(\mathcal{M})}^2 + \frac{1}{2} \kappa^2 C_p^2 \|\nabla \varsigma_h\|_{L^2(\Omega)}^2.$$

Let $\kappa = \min(C_p^{-2}, \gamma^{-1} C_t^{-2})$ to obtain

$$A_{UC}[(w_h, \varsigma_h), (w_h + \kappa \varsigma_h, -\varsigma_h)] \geq \frac{1}{2} \|w_h\|_{L^2(\mathcal{M})}^2 + \frac{1}{2} |w_h|_{s_1}^2 + \frac{\kappa}{2} \|\nabla \varsigma_h\|_{L^2(\Omega)}^2 \geq c |||(w_h, \varsigma_h)|||_{UC}^2.$$

In the last inequality the contribution $\|h w_h\|_{H^1(\Omega)}$ is added to the right-hand side and ς_h is controlled by applying (44). Using the triangle inequality and the right inequality of (42) we see that

$$|||(w_h + \kappa \varsigma_h, -\varsigma_h)|||_{UC} \leq |||(w_h, \varsigma_h)|||_{UC} + \|\kappa \varsigma_h\|_{H^1(\Omega)} + |\kappa \varsigma_h|_{s_1} \leq C |||(w_h, \varsigma_h)|||_{UC},$$

which concludes the proof. \square

Corollary 1. *The finite element formulation defined by (34) and (35), admits a unique solution.*

Proof. The immediate consequence of proposition 1. Since the system matrix is square we may consider (34) and (35), with the right-hand side equal to zero, and prove that the solution $(u_h, \lambda_h) = (0, 0)$ is unique. Assume that (u_h, λ_h) is a non-zero solution. Then by proposition 1,

$$c_s |||(u_h, \lambda_h)|||_{UC} \leq \sup_{(v_h, \mu_h) \in \mathcal{V}_h^{UC}} \frac{A_{UC}[(u_h, \lambda_h), (v_h, \mu_h)]}{|||(v_h, \mu_h)|||_{UC}} = 0,$$

which is a contradiction. \square

Proposition 2. *Let $(u_h, \lambda_h) \in \mathcal{V}_h^{UC}$ be the solution of (34), (35) and $u \in H^2(\Omega)$ be the solution to (10) and (11), with $\alpha = 0$. Then*

$$|||(u - u_h, \lambda_h)|||_{UC} \leq C(\|\delta u_0\|_{L^2(\mathcal{M})} + \|\delta f\|_{H^{-1}(\Omega)} + h|u|_{H^2(\Omega)}).$$

Proof. Let $\xi_h := u_h - i_h u$, with $i_h u$ denoting the Scott–Zhang interpolant of u . By the triangle inequality we have

$$|||(u - u_h, \lambda_h)|||_{UC} \leq |||(u - i_h u, 0)|||_{UC} + |||(\xi_h, \lambda_h)|||_{UC} \leq Ch|u|_{H^2(\Omega)} + |||(\xi_h, \lambda_h)|||_{UC}.$$

For the second term on the right-hand side we apply the inf-sup condition of proposition 1,

$$c_s |||(\xi_h, \lambda_h)|||_{UC} \leq \sup_{(v_h, \mu_h) \in \mathcal{V}_h^{UC}} \frac{A_{UC}[(\xi_h, \lambda_h), (v_h, \mu_h)]}{|||(v_h, \mu_h)|||_{UC}}. \quad (56)$$

Observing that under the regularity assumption on u , $s_1(u, v_h) = 0$ we have the consistency relation

$$\begin{aligned} A_{UC}[(u - u_h, -\lambda_h), (v_h, \mu_h)] &= m_\Omega(f - \tilde{f}, \mu_h) + m_{\mathcal{M}}(u_0 - \tilde{u}_0, v_h) \\ &= -m_\Omega(\delta f, \mu_h) - m_{\mathcal{M}}(\delta u_0, v_h) \end{aligned}$$

for all $(v_h, \mu_h) \in \mathcal{V}_h^{UC}$. We then obtain the equality

$$\begin{aligned} A_{UC}[(\xi_h, \lambda_h), (v_h, \mu_h)] &= A_{UC}[(\xi_h, \lambda_h), (v_h, \mu_h)] + A_{UC}[(u - u_h, -\lambda_h), (v_h, \mu_h)] \\ &\quad + m_{\mathcal{M}}(\delta u_0, v_h) + m_\Omega(\delta f, \mu_h) \\ &= A_{UC}[(u - i_h u, 0), (v_h, \mu_h)] + m_{\mathcal{M}}(\delta u_0, v_h) + m_\Omega(\delta f, \mu_h). \end{aligned}$$

Using the Cauchy–Schwarz inequality in the terms of the right-hand side we immediately deduce

$$\begin{aligned} A_{UC}[(\xi_h, \lambda_h), (v_h, \mu_h)] &\leq (\|\delta u_0\|_{L^2(\mathcal{M})} + \|\delta f\|_{H^{-1}(\Omega)} + \|\nabla(u - i_h u)\|_{L^2(\Omega)} + |||(u - i_h u, 0)|||_{UC}) \\ &\quad \times |||(v_h, \mu_h)|||_{UC}. \end{aligned}$$

Applying this inequality to (56) leads to the bound

$$c_s ||(\xi_h, \lambda_h)||_{UC} \leq \|\delta u_0\|_{L^2(\mathcal{M})} + \|\delta f\|_{H^{-1}(\Omega)} + \|\nabla(u - i_h u)\|_{L^2(\Omega)} + ||(u - i_h u, 0)||_{UC}$$

and the result follows from the approximation estimate (43) and (54). \square

Theorem 1. Let $(u_h, \lambda_h) \in \mathcal{V}_h^{UC}$ be the solution to (34) and (35) and $u \in H^2(\Omega)$ be the solution to (7) and (8). Then for some $0 < \tau < 1$ depending on $r_1/r_3, r_2/r_3$ there holds

$$\begin{aligned} \|u - u_h\|_{L^2(B_{r_2}(\mathbf{x}_0))} &\leq C_{h,\delta} (\|\delta u_0\|_{L^2(\mathcal{M})} + \|\delta f\|_{H^{-1}(\Omega)} + |u_h|_{s_1})^\tau \\ &\leq C_{h,\delta} (\|\delta u_0\|_{L^2(\mathcal{M})} + \|\delta f\|_{H^{-1}(\Omega)} + h|u|_{H^2(\Omega)})^\tau \end{aligned}$$

where

$$\begin{aligned} C_{h,\delta} &:= C(\|u - u_h\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} + |u_h|_{s_1} + \|\delta f\|_{H^{-1}(\Omega)})^{(1-\tau)} \\ &\leq C(h^{-1}\|\delta u_0\|_{L^2(\mathcal{M})} + h^{-1}\|\delta f\|_{H^{-1}(\Omega)} + \|u\|_{H^2(\Omega)})^{(1-\tau)}. \end{aligned}$$

Proof. Let $e = u - u_h$. First note that by a Poincaré inequality we have

$$\begin{aligned} \|e\|_{L^2(\Omega)} &\leq m_{\mathcal{M}}(e, e)^{\frac{1}{2}} + \|\nabla e\|_{L^2(\Omega)} \leq h^{-1} ||(e, 0)||_{UC} \\ &\leq C(h^{-1}\|\delta u_0\|_{L^2(\mathcal{M})} + h^{-1}\|\delta f\|_{H^{-1}(\Omega)} + \|u\|_{H^2(\Omega)}) \end{aligned} \quad (57)$$

where we used proposition 2 for the last inequality. This *a priori* estimate on the global error, together with the bound of proposition 2 on $|u_h|_{s_1}$ shows the upper bound on the coefficient $C_{h,\delta}$.

Injecting the error in the bilinear form $a(\cdot, w)$ we have for all $w \in H_0^1(\Omega)$,

$$a(e, w) = m_{\Omega}(f, w) - a(u_h, w) =: \langle r, w \rangle_{H^{-1}, H^1}, \text{ with } r \in H^{-1}(\Omega).$$

It follows that e is a weak solution to the problem (11) with the right-hand side $r \in H^{-1}(\Omega)$, and we are in the framework of lemma 1. Applying the result of the lemma to e we get, if $\|r\|_{H^{-1}(\Omega)} \leq \varepsilon$,

$$\|e\|_{L^2(B_{r_2})} \leq C \left(\|e\|_{L^2(B_{r_1})} + \varepsilon \right)^\tau \cdot \left(\|e\|_{L^2(B_{r_3})} + \varepsilon \right)^{(1-\tau)}. \quad (58)$$

Since $\|e\|_{L^2(B_{r_3})} \leq \|e\|_{L^2(\Omega)}$, which was bounded in (57) and $\|e\|_{L^2(B_{r_1})} \leq ||(e, 0)||_{UC}$ (since $B_{r_1} \subset \mathcal{M}$), which was bounded in proposition 2, we now only need to find ε such that

$$\|r\|_{H^{-1}(\Omega)} \leq \varepsilon \equiv \varepsilon(h, \delta u_0, \delta f, u)$$

and quantify the dependence of ε on $h, \delta u_0, \delta f$ and u . By definition of the dual norm we have

$$\|r\|_{H^{-1}(\Omega)} = \sup_{w \in H_0^1(\Omega) \setminus 0} \frac{\langle r, w \rangle_{H^{-1}, H^1}}{\|w\|_{H^1(\Omega)}}.$$

We proceed using the definition of r ,

$$\begin{aligned} \langle r, w \rangle_{H^{-1}, H^1} &= m_{\Omega}(f, w - i_h w) - a(u_h, w - i_h w) - m_{\Omega}(\delta f, i_h w) \\ &\leq C_r(h\|f\|_{L^2(\Omega)} + |u_h|_{s_1} + \|\delta f\|_{H^{-1}(\Omega)})\|w\|_{H^1(\Omega)}. \end{aligned}$$

Here we used a partial integration in the form $a(\cdot, \cdot)$, the first trace inequality of (41) and the approximation to obtain

$$|a(u_h, w - i_h w)| \leq \sum_{F \in \mathcal{F}_i} \int_F \|\nabla u_h \cdot n_F\| |w - i_h w| \, ds \leq C |u_h|_{s_1} \|w\|_{H^1(\Omega)}.$$

We deduce that $\|r\|_{H^{-1}(\Omega)} \leq \varepsilon$ holds for $\varepsilon = C_r(h\|f\|_{L^2(\Omega)} + |u_h|_{s_1} + \|\delta f\|_{H^{-1}(\Omega)})$ which concludes the proof. \square

Corollary 2. Assume that for $h_0 > 0$ there holds

$$\|\delta u_0\|_{L^2(\mathcal{M})} + \|\delta f\|_{H^{-1}(\Omega)} \leq h_0 \|u\|_{H^2(\Omega)} \quad (59)$$

then for $h > h_0$, there exists C_0 such that

$$\|u - u_h\|_{L^2(B_{r_2}(\mathbf{x}_0))} \leq C_0 h^\tau \|u\|_{H^2(\Omega)}$$

with C_0 independent of h .

Proof. By the assumed bound (59) we have for $h > h_0$,

$$(h^{-1} \|\delta u_0\|_{L^2(\mathcal{M})} + h^{-1} \|\delta f\|_{H^{-1}(\Omega)} + \|u\|_{H^2(\Omega)})^{(1-\tau)} \leq (2\|u\|_{H^2(\Omega)})^{(1-\tau)}$$

and

$$(\|\delta u_0\|_{L^2(\mathcal{M})} + \|\delta f\|_{H^{-1}(\Omega)} + h\|u\|_{H^2(\Omega)})^\tau \leq h^\tau (2\|u\|_{H^2(\Omega)})^\tau.$$

The claim follows by using these bounds in the error estimate of theorem 1. \square

Observe that theorem 1 provides both *a priori* and *a posteriori* error bounds. This means that perturbations in the data can be compared with the computational residual in an *a posteriori* procedure to drive adaptive algorithms for the computation of the reconstruction.

5. Error analysis—source reconstruction

The error analysis in this case follows a similar outline, however instead of lemma 1 we may here use a discrete interpolation argument to obtain convergence orders.

We introduce the triple norm

$$|||(v_h, w_h, \mu_h)|||_{SR} := \|v_h\|_{L^2(\Omega)} + \|hw_h\|_{L^2(\Omega)} + \|h^{-1}\mu_h\|_{L^2(\Omega)} + |v_h|_{s_1} + |w_h|_{s_5}, \quad (60)$$

where we recall that

$$|x_h|_{s_i} := s_i(x_h, x_h)^{\frac{1}{2}}.$$

We will also use the H^1 -projection defined by $\Pi_h u \in V_h^0$ such that

$$a(\Pi_h u, v_h) = a(u, v_h), \quad \forall v_h \in V_h^0.$$

It is well known that if Ω is convex the following estimate holds

$$\|u - \Pi_h u\|_{L^2(\Omega)} + h\|\nabla(u - \Pi_h u)\|_{L^2(\Omega)} \leq Ch^2|u|_{H^2(\Omega)}.$$

We will first prove an estimate where we assume that q is more regular and show that in this case the stabilization of the velocity is superfluous.

Proposition 3. *Let $(u, q) \in H_0^1(\Omega) \times H^1(\Omega)$ satisfy (20), (21) and let $(u_h, q_h, \lambda_h) \in \mathcal{V}_h^{SR}$ be the solution of (38)–(40), with $\gamma_1 = 0$ and $\gamma_5 \geq 0$. Then there holds*

$$\|u - u_h\|_{L^2(\Omega)} + h\|\nabla(u - u_h)\|_{L^2(\Omega)} + |\pi_h q - q_h|_{s_5} \leq C(h^2(1 + \gamma_5^{\frac{1}{2}})\|q\|_{H^1(\Omega)} + \|\delta u\|_{L^2(\Omega)}). \quad (61)$$

For $\gamma_5 \geq 0$

$$\|\pi_h^0(q - q_h)\|_{H^{-1}(\Omega)} \leq C(h(1 + \gamma_5^{\frac{1}{2}})\|q\|_{H^1(\Omega)} + h^{-1}\|\delta u\|_{L^2(\Omega)}) \quad (62)$$

and for $\gamma_5 > 0$

$$\|q - q_h\|_{H^{-1}(\Omega)} \leq C(h(1 + \gamma_5^{\frac{1}{2}})\|q\|_{H^1(\Omega)} + h^{-1}\|\delta u\|_{L^2(\Omega)}). \quad (63)$$

Proof. Let $\xi_h = u_h - \Pi_h u$ and $\eta_h = q_h - \pi_h q$. It follows by the definition of $A_h[(\cdot, \cdot, \cdot), (\cdot, \cdot, \cdot)]$ that

$$\|\xi_h\|_{L^2(\Omega)}^2 + |\eta_h|_{s_5}^2 = A_{SR}[(\xi_h, \eta_h, \lambda_h), (\xi_h, \eta_h, \lambda_h)].$$

By the definition of (38)–(40) and using that (u, q) satisfy (20), we may write

$$A_{SR}[(u_h, q_h, \lambda_h), (\xi_h, \eta_h, \lambda_h)] = m_\Omega(\tilde{u}_0, \xi_h) - a(u, \lambda_h) + m_\Omega(q, \lambda_h).$$

It follows that

$$\begin{aligned} \|\xi_h\|_{L^2(\Omega)}^2 + |\eta_h|_{s_5}^2 &= A_{SR}[(\xi_h, \eta_h, \lambda_h), (\xi_h, \eta_h, \lambda_h)] \\ &= A_{SR}[(u_h, q_h, \lambda_h), (\xi_h, \eta_h, \lambda_h)] - A_{SR}[(\Pi_h u, \pi_h q, 0), (\xi_h, \eta_h, \lambda_h)] \\ &= m_\Omega(\tilde{u}_0 - \Pi_h u, \xi_h) - s_5(\pi_h q, \eta_h) - a(u - \Pi_h u, \lambda_h) + m_\Omega(q - \pi_h q, \lambda_h) \\ &= m_\Omega(\tilde{u}_0 - \Pi_h u, \xi_h) - s_5(\pi_h q, \eta_h) \end{aligned}$$

where the last equality follows by the orthogonality properties of the H^1 – and L^2 – projectors. Using a Cauchy–Schwarz inequality we then obtain

$$\|\xi_h\|_{L^2(\Omega)}^2 + |\eta_h|_{s_5}^2 \leq (\|u - \Pi_h u\|_{L^2(\Omega)} + \|\delta u\|_{L^2(\Omega)} + |\pi_h q|_{s_5})(\|\xi_h\|_{L^2(\Omega)} + |\eta_h|_{s_5}).$$

Using the approximation properties of the Ritz projection, the right relation of (42) and the H^1 -stability of the L^2 -projection we get the estimate

$$\|\xi_h\|_{L^2(\Omega)} + |\eta_h|_{s_5} \leq Ch^2(|u|_{H^2(\Omega)} + \gamma_5\|\nabla q\|_{L^2(\Omega)}) + \|\delta u\|_{L^2(\Omega)}. \quad (64)$$

The estimate on the gradient of the error is then the consequence of an inverse inequality

$$\|\nabla \xi_h\|_{L^2(\Omega)} \leq C_\iota h^{-1}\|\xi_h\|_{L^2(\Omega)} \leq Ch(|u|_{H^2(\Omega)} + \|\nabla q\|_{L^2(\Omega)}) + h^{-1}\|\delta u\|_{L^2(\Omega)}. \quad (65)$$

Using the triangle inequality we have

$$\|u - u_h\|_{L^2(\Omega)} + h\|\nabla(u - u_h)\|_{L^2(\Omega)} \leq \|u - \Pi_h u\|_{L^2(\Omega)} + h\|\nabla(u - \Pi_h u)\|_{L^2(\Omega)} + \|\xi_h\|_{L^2(\Omega)} + h\|\nabla \xi_h\|_{L^2(\Omega)}$$

and (61) follows from (64) and (65). For the estimate (62) on the source term observe that for $\gamma_5 \geq 0$ we may use the orthogonality $m_\Omega(\pi_h^0(q - q_h), w - \pi_h^0 w) = 0$ to obtain the bound

$$\begin{aligned} \|\pi_h^0(q - q_h)\|_{H^{-1}(\Omega)} &= \sup_{w \in H_0^1(\Omega): \|w\|_{H^1}=1} m_\Omega(\pi_h^0(q - q_h), w - \pi_h^0 w) + a(u - u_h, \pi_h^0 w) \\ &\leq C\|\nabla(u - u_h)\|_{L^2(\Omega)} \end{aligned} \quad (66)$$

from which (62) follows using (61). If on the other hand $\gamma_5 > 0$, then we may use

$$\|q - q_h\|_{H^{-1}(\Omega)} \leq \|q - \pi_h q\|_{H^{-1}(\Omega)} + \|\pi_h q - q_h\|_{H^{-1}(\Omega)},$$

where it is immediately shown that $\|q - \pi_h q\|_{H^{-1}(\Omega)} \leq Ch^2\|\nabla q\|_{L^2(\Omega)}$ and

$$\|\pi_h q - q_h\|_{H^{-1}(\Omega)} \leq \|(\pi_h q - q_h) - \pi_h^0(\pi_h q - q_h)\|_{H^{-1}(\Omega)} + \|\pi_h^0(q - q_h)\|_{H^{-1}(\Omega)}.$$

For the second term on the right-hand side the estimate (66) holds and for the first term we observe that with $\eta_h - \pi_h^0 \eta_h = (\pi_h q - q_h) - \pi_h^0(\pi_h q - q_h)$ we have

$$\begin{aligned} \|\eta_h - \pi_h^0 \eta_h\|_{H^{-1}(\Omega)} &= \sup_{w \in H_0^1(\Omega): \|w\|_{H^1}=1} m_\Omega(\eta_h - \pi_h^0 \eta_h, w - \pi_h^0 w) \\ &\leq Ch\|\eta_h - \pi_h^0 \eta_h\|_{L^2(\Omega)} \leq C(|\eta_h|_{s_5} + h\|\pi_h^0 \eta_h\|_{L^2(\Omega)}). \end{aligned}$$

Here lemma 2 (equation (45)) was used for the last inequality. We may then use the equation to deduce

$$h^2\|\pi_h^0 \eta_h\|_{L^2(\Omega)}^2 = h^2 m_\Omega(q - q_h, \pi_h^0 \eta_h) = h^2 a(u - u_h, \pi_h^0 \eta_h)$$

and after a Cauchy–Schwarz inequality and an inverse inequality (third inequality of (41)) in the second factor,

$$h^2 a(u - u_h, \pi_h^0 \eta_h) \leq Ch\|\nabla(u - u_h)\|_{L^2(\Omega)}\|\pi_h^0 \eta_h\|_{L^2(\Omega)}.$$

It follows that $h\|\pi_h^0 \eta_h\|_{L^2(\Omega)} \leq C\|\nabla(u - u_h)\|_{L^2(\Omega)}$. We conclude that the inequality (63) holds using the above bounds together with (61). \square

We see that for smooth source terms we can also expect relatively good behavior when standard regularization terms are used. However, if the source term is less regular, i.e. $q \in L^2(\Omega)$, then convergence (with an order in h) can no longer be deduced. In order to obtain an estimate with the convergence order in h in the case where $q \in L^2(\Omega)$ as well, we take $\gamma_1 > 0$ and prove that this allows us to obtain stronger control of the approximation of the source term, leading to the desired estimate.

Proposition 4 (Inf-sup stability). *Let $\gamma_1, \gamma_5 > 0$. There exists $c_s > 0$ such that for all $(y_h, t_h, s_h) \in \mathcal{V}_h^{SR}$ there holds*

$$c_s \| |(y_h, t_h, s_h)| \|_{SR} \leq \sup_{(v_h, w_h, \mu_h) \in \mathcal{V}_h^{SR}} \frac{A_{SR}[(y_h, t_h, s_h), (v_h, w_h, \mu_h)]}{\| |(v_h, w_h, \mu_h)| \|_{SR}}.$$

Proof. For some $\kappa > 0$ to be fixed, take $v_h = y_h$, $w_h = t_h - \kappa h^{-2} \varsigma_h$, $\mu_h = \varsigma_h + \kappa h^2 \pi_h^0 t_h$ to obtain

$$\begin{aligned} A_{SR}[(y_h, t_h, \varsigma_h), (y_h, t_h - \kappa h^{-2} \varsigma_h, \varsigma_h + \kappa h^2 \pi_h^0 t_h)] &= |y_h|_{s_1}^2 + |t_h|_{s_5}^2 + \|y_h\|_{L^2(\Omega)}^2 \\ &\quad + \kappa \|h^{-1} \varsigma_h\|_{L^2(\Omega)}^2 + \kappa \|h \pi_h^0 t_h\|_{L^2(\Omega)}^2 \\ &\quad - s_5(t_h, \kappa h^{-2} \varsigma_h) - a(y_h, \kappa h^2 \pi_h^0 t_h). \end{aligned}$$

The Cauchy–Schwarz inequality, the arithmetic–geometric inequality and the stability inequality (42), with $i = 5$ lead to

$$s_5(t_h, \kappa h^{-2} \varsigma_h) \leq \frac{1}{2} |t_h|_{s_5}^2 + \frac{1}{2} C_{st}^2 \kappa^2 \|h^{-1} \varsigma_h\|_{L^2(\Omega)}^2.$$

Using partial integration, the fact that $\pi_h^0 t_h|_{\partial\Omega} = 0$ and an elementwise trace inequality (41), we have the bound

$$a(y_h, \kappa h^2 \pi_h^0 t_h) \leq \frac{1}{2} |y_h|_{s_1}^2 + \frac{1}{2} C_t^2 \kappa^2 \|h \pi_h^0 t_h\|_{L^2(\Omega)}^2.$$

We may then fix $\kappa = \min(C_{st}^{-2}, C_t^{-2})$ to show that for some $c > 0$ depending only on the mesh geometry

$$\begin{aligned} c(|y_h|_{s_1}^2 + |t_h|_{s_5}^2 + \|y_h\|_{L^2(\Omega)}^2 + \kappa \|h^{-1} \varsigma_h\|_{L^2(\Omega)}^2 + \kappa \|h \pi_h^0 t_h\|_{L^2(\Omega)}^2) \\ \leq A_{SR}[(y_h, t_h, \varsigma_h), (y_h, t_h - \kappa h^{-2} \varsigma_h, \varsigma_h + \kappa h^2 \pi_h^0 t_h)]. \end{aligned}$$

By lemma 2 and the quasi-uniformity of the mesh, there holds for some $c > 0$ depending only on the mesh geometry

$$c \|h t_h\|_{L^2(\Omega)}^2 \leq \|h \pi_h^0 t_h\|_{L^2(\Omega)}^2 + |t_h|_{s_5}^2$$

and it follows that for some $c > 0$ depending only on the mesh geometry

$$c |||(y_h, t_h, \varsigma_h)|||_{SR}^2 \leq A_{SR}[(y_h, t_h, \varsigma_h), (y_h, t_h - \kappa h^{-2} \varsigma_h, \varsigma_h + \kappa h^2 \pi_h^0 t_h)].$$

To conclude we need to show that

$$|||(y_h, t_h - \kappa h^{-2} \varsigma_h, \varsigma_h + \kappa h^2 \pi_h^0 t_h)|||_{SR} \leq C |||(y_h, t_h, \varsigma_h)|||_{SR}.$$

To this end we note that

$$|||(y_h, t_h - \kappa h^{-2} \varsigma_h, \varsigma_h + \kappa h^2 \pi_h^0 t_h)|||_{SR} \leq |||(y_h, t_h, \varsigma_h)|||_{SR} + \kappa |||(0, h^{-2} \varsigma_h, h^2 \pi_h^0 t_h)|||_{SR}.$$

For the second term on the right-hand side we use (42) (the left inequality) to write

$$\begin{aligned} |||(0, h^{-2} \varsigma_h, h^2 \pi_h^0 t_h)|||_{SR}^2 &= \|h(h^{-2} \varsigma_h)\|_{L^2(\Omega)}^2 + \|h^{-1}(h^2 \pi_h^0 t_h)\|_{L^2(\Omega)}^2 + |h^{-2} \varsigma_h|_{s_5}^2 \\ &\leq C(\|h^{-1} \varsigma_h\|_{L^2(\Omega)}^2 + \|h t_h\|_{L^2(\Omega)}^2) \leq C |||(y_h, t_h, \varsigma_h)|||_{SR}^2. \end{aligned}$$

The constant only depends on the constant of quasi-uniformity of the meshes. \square

Theorem 2. Let $(u, q) \in H_0^1(\Omega) \cap H^2(\Omega) \times L^2(\Omega)$ satisfy (20) and let $(u_h, q_h, \lambda_h) \in \mathcal{V}_h^{SR}$ be the solution of (38)–(40). Then there holds

$$|||(u - u_h, q - q_h, \lambda_h)|||_{SR} \leq C(h\|q\|_{L^2(\Omega)} + \|\delta u\|_{L^2(\Omega)})$$

and

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} + \|q - q_h\|_{H^{-1}(\Omega)} \leq C(h^{\frac{1}{2}}\|q\|_{L^2(\Omega)} + h^{-\frac{1}{2}}\|\delta u\|_{L^2(\Omega)}).$$

Proof. As before, using a triangle inequality and an approximation, it is enough to consider the discrete errors $\xi_h = u_h - \Pi_h u$ and $\eta_h = q_h - \pi_h q$. From proposition 4 we know that

$$c_s |||(\xi_h, \eta_h, \lambda_h)|||_{SR} \leq \sup_{(v_h, w_h, \mu_h) \in \mathcal{V}_h^{SR}} \frac{A_{SR}[(\xi_h, \eta_h, \lambda_h), (v_h, w_h, \mu_h)]}{|||(v_h, w_h, \mu_h)|||_{SR}}.$$

Using the definition of (20) and (38)–(40) we may write

$$\begin{aligned} A_{SR}[(\xi_h, \eta_h, \lambda_h), (v_h, w_h, \mu_h)] &= A_{SR}[(u_h, q_h, \lambda_h), (v_h, w_h, \mu_h)] - A_{SR}[(\Pi_h u, \pi_h q, \lambda_h), (v_h, w_h, \mu_h)] \\ &= m_\Omega(\tilde{u}_0 - \Pi_h u, v_h) - s_1(\Pi_h u, v_h) \\ &\quad - s_5(\pi_h q, w_h) - a(u - \Pi_h u, \mu_h) + m_\Omega(q - \pi_h q, \mu_h) \\ &= I + II + III + IV + V. \end{aligned}$$

We see that using a Cauchy–Schwarz inequality, the approximation properties of the H^1 -projection and the regularity of u , we have under the assumed regularity bound $|u|_{H^2(\Omega)} \leq C\|q\|_{L^2(\Omega)}$,

$$I \leq (\|u - \Pi_h u\|_{L^2(\Omega)} + \|\delta u\|_{L^2(\Omega)})\|v_h\|_{L^2(\Omega)} \leq C(h^2\|q\|_{L^2(\Omega)} + \|\delta u\|_{L^2(\Omega)})\|v_h\|_{L^2(\Omega)}.$$

Similarly for term II we use the Cauchy–Schwarz inequality and arguments identical to the approximation result (55) to obtain

$$II \leq s_1(u - \Pi_h u, u - \Pi_h u)^{\frac{1}{2}} s_1(v_h, v_h)^{\frac{1}{2}} \leq Ch\|q\|_{L^2(\Omega)}|v_h|_{s_1}.$$

In term III we use the Cauchy–Schwarz inequality, the left relation of (42) and the stability of the L^2 -projection to obtain

$$III \leq s_5(\pi_h q, \pi_h q)^{\frac{1}{2}} s_5(w_h, w_h)^{\frac{1}{2}} \leq Ch\|q\|_{L^2(\Omega)}|w_h|_{s_5}.$$

Finally for term IV and V we use the orthogonality of the projections to deduce that they are zero. Collecting the bounds for the terms $I - V$ above we have shown the following bound:

$$\begin{aligned} A_{SR}[(\xi_h, \eta_h, \lambda_h), (v_h, w_h, \mu_h)] &\leq C(h\|q\|_{L^2(\Omega)} + \|\delta u\|_{L^2(\Omega)}) \\ &\quad \times (\|v_h\|_{L^2(\Omega)}^2 + |w_h|_{s_5}^2 + |v_h|_{s_1}^2)^{\frac{1}{2}} \\ &\leq C(h\|q\|_{L^2(\Omega)} + \|\delta u\|_{L^2(\Omega)})|||(v_h, w_h, \mu_h)|||_{SR}, \end{aligned}$$

which proves the first claim.

The second claim follows from the first using the added stability of the velocity stabilization $s_1(\cdot, \cdot)$. By the triangle inequality there holds

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \leq \|\nabla(u - \Pi_h u)\|_{L^2(\Omega)} + \|\nabla \xi_h\|_{L^2(\Omega)} \leq Ch\|q\|_{L^2(\Omega)} + \|\nabla \xi_h\|_{L^2(\Omega)}.$$

Using an integration by parts, followed by the Cauchy–Schwarz inequality, an elementwise trace inequality (41) and the first claim of the theorem we see that

$$\begin{aligned} \|\nabla \xi_h\|_{L^2(\Omega)}^2 &\leq \sum_{F \in \mathcal{F}_I} \frac{1}{2} \int_F \|\llbracket \nabla \xi_h \cdot n_F \rrbracket\| |\xi_h| \, ds \\ &\leq \gamma_1^{-\frac{1}{2}} |\xi_h|_{s_1} h^{-1} C_t \|\xi_h\|_{L^2(\Omega)} \leq \gamma_1^{-\frac{1}{2}} h^{-1} \|(\xi_h, 0, 0)\|_{SR}^2 \\ &\leq \gamma_1^{-\frac{1}{2}} h^{-1} C(h\|q\|_{L^2(\Omega)} + \|\delta u\|_{L^2(\Omega)})^2. \end{aligned}$$

Taking the square roots of both sides we see that

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \leq C(h^{\frac{1}{2}}\|q\|_{L^2(\Omega)} + h^{-\frac{1}{2}}\|\delta u\|_{L^2(\Omega)}). \quad (67)$$

The result on $\|q - q_h\|_{H^{-1}(\Omega)}$ follows using a duality argument, the formulation (38)–(40) and the standard approximation (43)

$$\begin{aligned} \|q - q_h\|_{H^{-1}(\Omega)} &:= \sup_{w \in H_0^1(\Omega): \|w\|_{H^1(\Omega)}=1} m_\Omega(q - q_h, w) \\ &= \sup_{w \in H_0^1(\Omega): \|w\|_{H^1(\Omega)}=1} (m_\Omega(q - q_h, w - i_h w) - a(u - u_h, i_h w)) \\ &\leq C(\|h(q - q_h)\|_{L^2(\Omega)} + \|\nabla(u - u_h)\|_{L^2(\Omega)}). \end{aligned}$$

The gradient was bounded in (67) and the L^2 -contribution is bounded using the first claim. This concludes the proof. \square

6. Numerical examples

6.1. Unique continuation

We consider the domain $\Omega = (0, 1) \times (0, 1)$ and the exact solution

$$u(x, y) = \frac{1}{(1 + \epsilon - x)(\epsilon + x)(1 + \epsilon - y)(\epsilon + y)}$$

and $f = -\Delta u$. As ϵ becomes small, a layer is created on the boundary. We let the data domain be defined by

$$\mathcal{M} := \{(x, y) \in \Omega : |x - 0.5| \leq 0.15 \text{ and } |y - 0.5| \leq 0.15\}$$

and consider the following subsets of Ω for the error studies:

$$\omega_1 := \{(x, y) \in \Omega : |x - 0.5| \leq 0.25 \text{ and } |y - 0.5| \leq 0.25\}$$

$$\omega_2 := \{(x, y) \in \Omega : |x - 0.5| \leq 0.35 \text{ and } |y - 0.5| \leq 0.35\}.$$

When Tikhonov regularization is used we choose the regularization parameter that is optimal for $\tau = 0.5$, i.e. $\alpha = \gamma_\alpha h^{8/3}$, where γ is a free parameter. According to our analysis above, for $\tau = 0.5$, this choice leads to a convergence of $O(h^{0.4})$ instead of $O(h^{0.5})$, which is the corresponding value for the stabilized finite element method. We therefore expect the local quantities to have comparable convergence properties for both methods. We tune the stabilization parameter γ_1 in the stabilized finite element method or γ_α for the regularized method by performing a series of computations with $\epsilon = 0.05$, taking $\gamma = 10^{-k}$, $k \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

on a mesh with 40 elements on the side. The result is presented in figure 1. Here we present the global relative L^2 -error plotted against the stabilization parameter—but the behavior was also the same for the local L^2 -errors over ω_1 and ω_2 . We see that the optimal parameter choice is $\gamma_\alpha = 10^{-4}$ and $\gamma_1 = 10^{-7}$. We also observe that the error of the stabilized finite element method is more well behaved under variation of the parameter with a lower level of error on the plateau away from the sweet spot. We computed the solution for $\epsilon = 0.05$ and $\epsilon = 0.15$ on a sequence of criss-cross meshes with the number of elements on each side taken in the set $\{40, 80, 160, 320, 640\}$. To avoid spurious effects due to using the ‘sweet spot’ parameter value on a given mesh, we used the optimal parameters from the test above multiplied by 100, i.e. $\gamma_1 = 10^{-5}$ and $\gamma_\alpha = 10^{-2}$. In figures 2 and 3 we juxtapose the solution of the stabilized finite element method (left plot) with the one using Tikhonov regularization (right plot) for $\epsilon = 0.05$ and $\epsilon = 0.15$ respectively. The error quantities are represented by the dashed lines and distinguished by markers according to the following scheme:

- (i) \square : the relative global H^1 -semi-norm;
- (ii) \circ : the relative global L^2 -norm;
- (iii) \diamond : the relative $L^2(\omega_2)$ -norm;
- (iv) ∇ : the relative $L^2(\omega_1)$ -norm;
- (v) \triangle : the relative $L^2(\mathcal{M})$ -norm;

To be able to compare the graphs we also add four dotted lines that are reference slopes corresponding to (from top to bottom) $x^{-0.5}$, $x^{-0.75}$, $0.5x^{-1}$, $4x^{-2}$ respectively. We make the following observations:

- (i) The expected convergence rates are observed: $O(h^\tau)$ with $\tau \in (0.5, 1)$ for the local quantities and inverse logarithmic for the global quantities.
- (ii) As predicted by the theory, the convergence rates of the local L^2 -norms appear to be slightly better for the stabilized method, and the constant is larger for the computations using Tikhonov regularization.
- (iii) The convergence of the global error (inverse logarithmic) can be observed for the stabilized finite element method, with best relative errors of 2% ($\epsilon = 0.15$) and 15% ($\epsilon = 0.05$) for the L^2 -norm and 10% ($\epsilon = 0.15$) and 20% ($\epsilon = 0.05$) for the H^1 -norm. The convergence of the global errors is very poor for Tikhonov FEM, with the best relative errors of 12% ($\epsilon = 0.15$) and 40% ($\epsilon = 0.05$) for the L^2 -norm and 44% ($\epsilon = 0.15$) and 50% ($\epsilon = 0.05$) for the H^1 -norm.
- (iv) When comparing similar error quantities the error in the stabilized finite element solution is always smaller than that of the Tikhonov FEM solution. In results not reported here we verified that even if both methods are used with the parameter value 10^{-4} (i.e. the optimal parameter for the Tikhonov FEM), the stabilized finite element method reaches smaller errors on sufficiently fine meshes for all error quantities, thanks to its better convergence and stagnation for lower values of the error.

6.2. Source reconstruction

6.2.1. Convergence for smooth and non-smooth sources, with perturbation of data. Again, we consider the domain $\Omega = (-1, 1) \times (-1, 1)$ and the data $u_0 = (x + 1)(x - 1)(y + 1)(y - 1)$, corresponding to the smooth source term $q = 2(2 - x^2 - y^2)$.

In figure 4 we show the interpolant of the exact source and a typical solution obtained using the gradient jump stabilization method. In figure 5 we show the effect of (properly scaled)

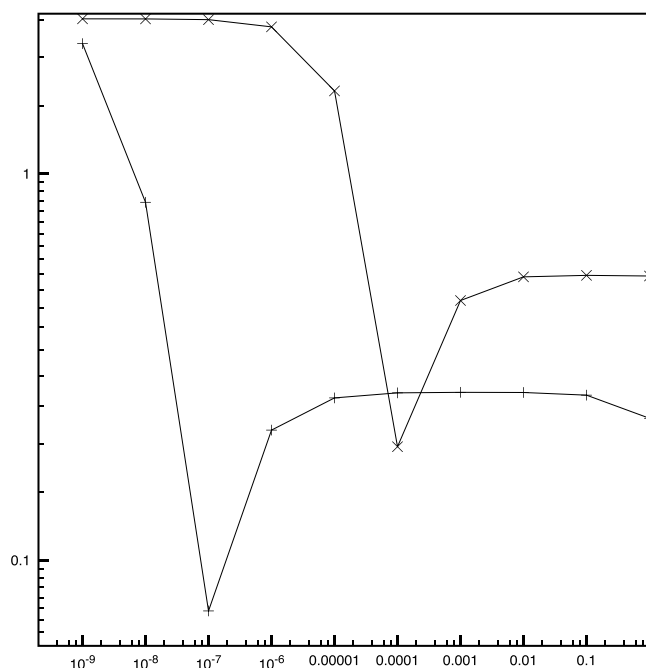


Figure 1. The sensitivity of the global L^2 -error (ordinate) with respect to the variation of the stabilization parameter γ (abscissa) on the mesh with 40 elements on the side; '+' indicates the stabilized FEM (γ_1) and 'x' indicates the Tikhonov FEM (γ_α).

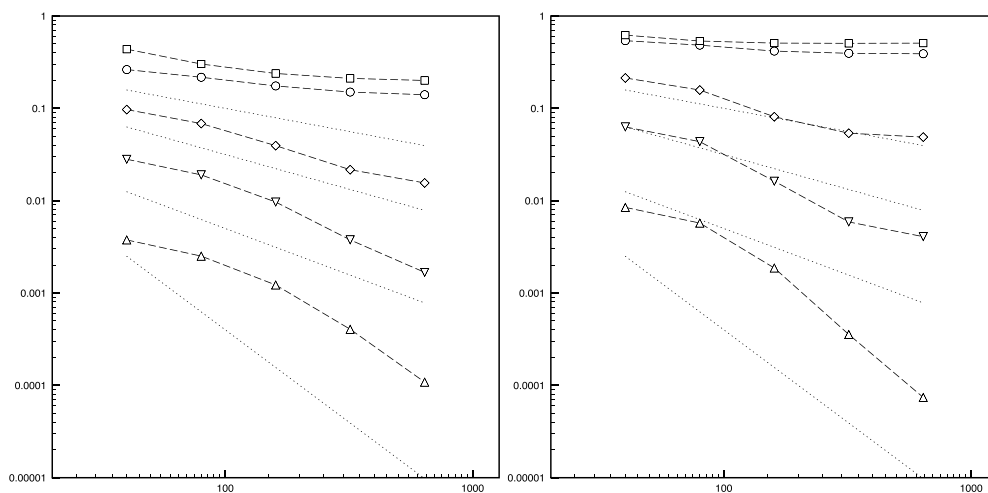


Figure 2. Left: various errors of the stabilized FEM; right: various errors of the Tikhonov FEM. $\epsilon = 0.05$; on the abscissa number of elements on the side of the domain. The stabilization parameter is $100\times$ the optimal choice ($\gamma_1 = 10^{-5}$, $\gamma_\alpha = 10^{-2}$). The dotted lines reference the slopes (from top to bottom) $x^{-0.5}$, $x^{-0.75}$, $0.5x^{-1}$ and $4x^{-2}$.

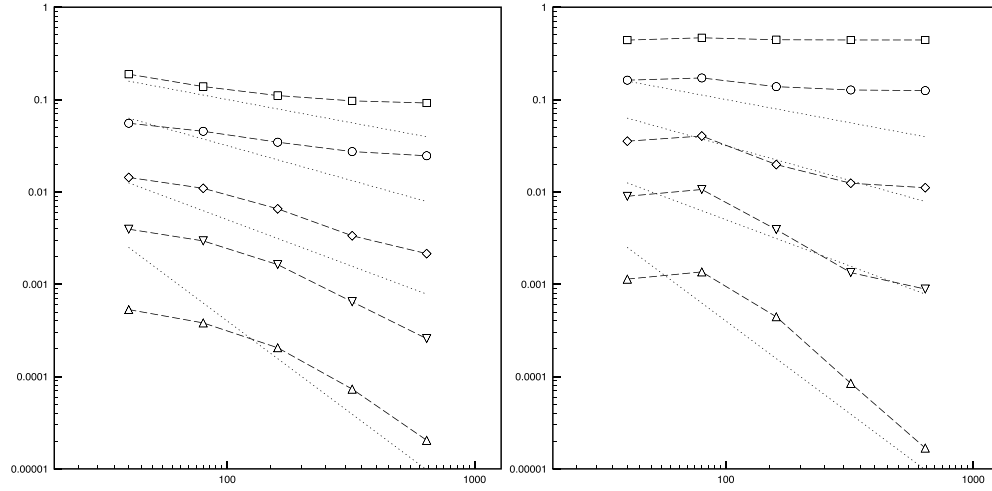


Figure 3. Left: various errors of the stabilized FEM; right: various errors of the Tikhonov FEM. $\epsilon = 0.15$; on the abscissa number of elements on the side of the domain. The stabilization parameter is $100 \times$ the optimal choice ($\gamma_1 = 10^{-5}$, $\gamma_\alpha = 10^{-2}$). The dotted lines reference the slopes (from top to bottom) $x^{-0.5}$, $x^{-0.75}$, $0.5x^{-1}$ and $4x^{-2}$.

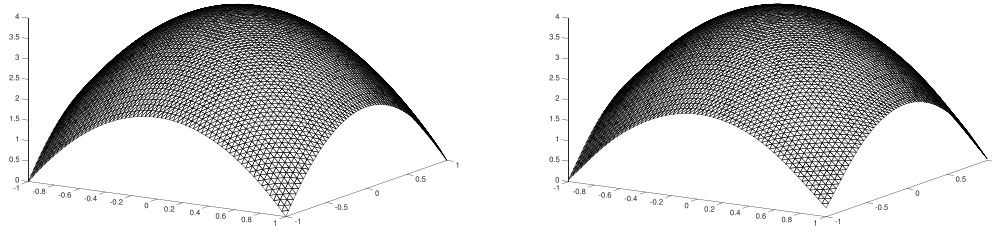


Figure 4. Left: the interpolated source term; right: the source term obtained with the present method.

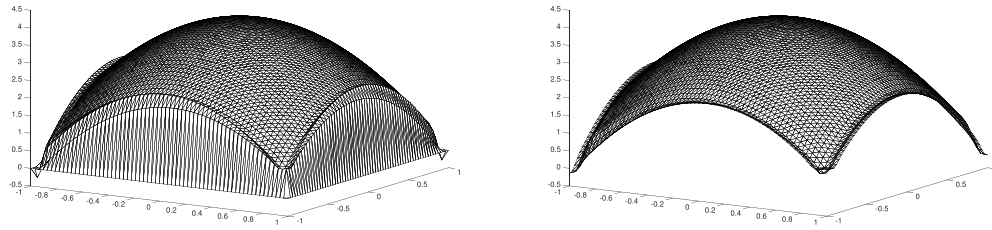


Figure 5. Left: the Tikhonov regularization with zero order term; right: the Tikhonov regularization with the Laplacian.

Tikhonov regularization using L_2 , i.e. $\|q\|_{L^2(\Omega)}$, and H^1 , i.e. $\|\nabla q\|_{L^2(\Omega)}$, regularizations, respectively. Note that the L_2 regularization gives the wrong boundary conditions in the discrete scheme, whereas H^1 works better while still giving a spurious boundary effect, which is well known from similar approaches used in fluid mechanics, see Burman and Hansbo [13].

The observed convergence for the method (38)–(40) using only the stabilization term $s_5(q_h, w_h)$ and for a variety of choices of γ is shown in figure 6. We note that the convergence

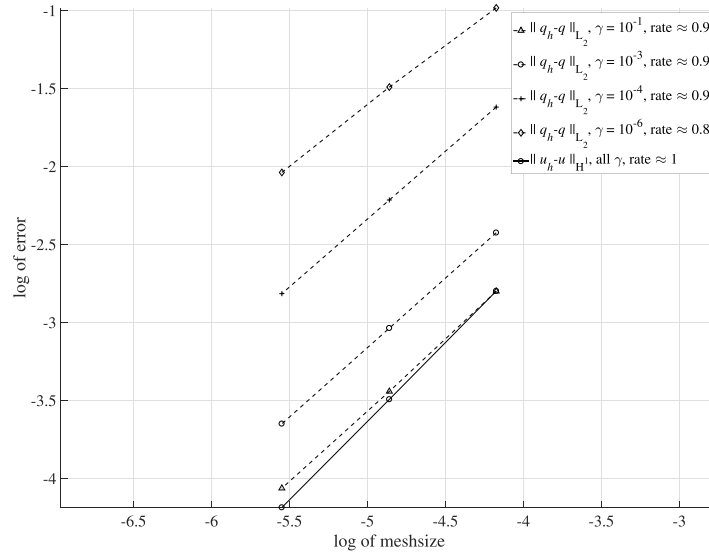


Figure 6. The convergence for a smooth source term.

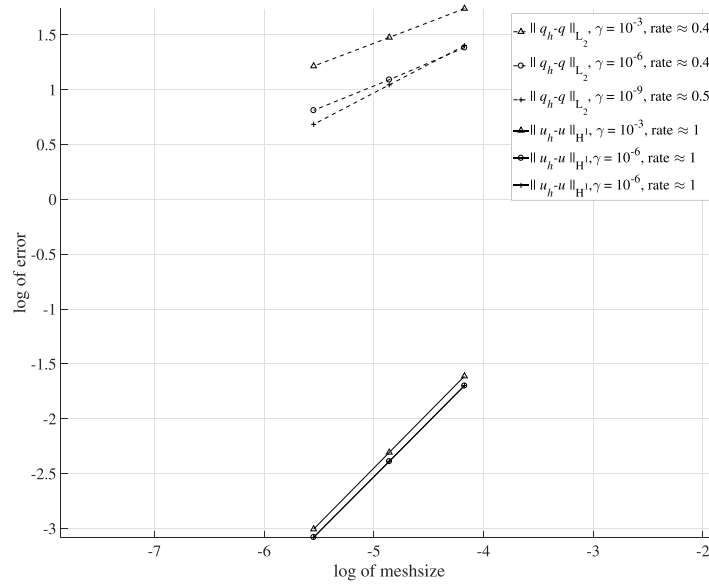


Figure 7. The convergence for a non-smooth source term.

$\|q_h - q\|_{L_2(\Omega)}$ and $\|u_h - u\|_{H^1(\Omega)}$ is of first order in both cases. The convergence of u_h is completely unaffected by the choice of γ .

For the non-smooth case, we let the solution be two different constants in the radial direction, $u = 1$ for $0 \leq r \leq 1/4$ and $u = 0$ for $3/4 \leq r$, interconnected by a cubic C^1 -polynomial in the radial direction. This means that the source term will have jumps at $r = 1/4$ and $r = 3/4$ so that $q \in H^{1/2-\epsilon}(\Omega)$ for any $\epsilon > 0$. In figure 7 we show the observed rate of convergence,

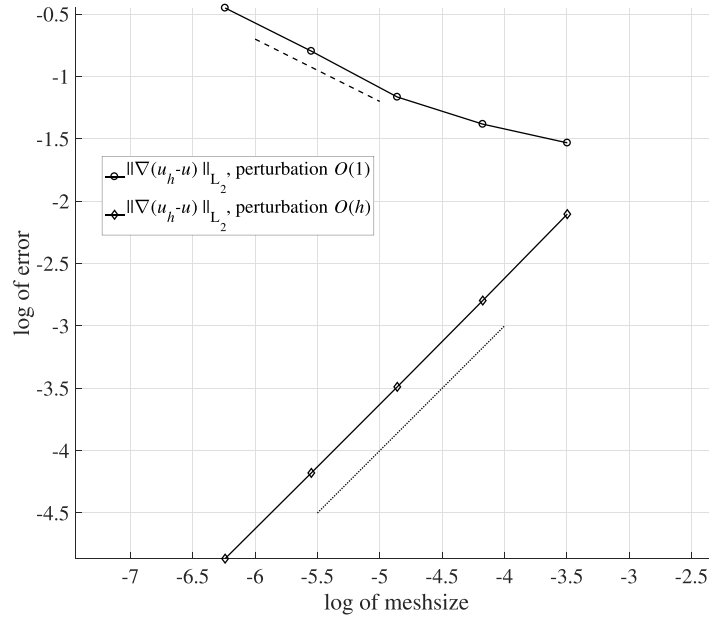


Figure 8. The convergence for a smooth source with perturbed data u_0 . The dashed line has the inclination $-1/2$, the dotted line has the inclination $1/1$.

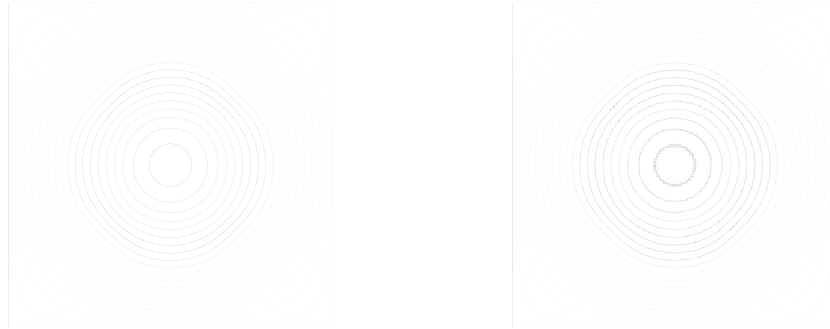


Figure 9. Left: the data u_0 using the fitted source term. Right: the data u_0 using the unfitted source term.

which drops to about $O(h^{1/2})$ for $\|q_h - q\|_{L_2(\Omega)}$ but remains $O(h)$ for $\|u_h - u\|_{H^1(\Omega)}$. The error constant is now also affected by γ for the convergence of u_h .

Finally, we show the effect of perturbing the data randomly, with a constant amplitude and with the amplitude decreasing $O(h)$. In figure 8 we show the obtained convergence in the H^1 -semi-norm. The convergence is $O(h^{-1/2})$ and $O(h)$, respectively, see theorem 2.

6.2.2. Measurement error. Consider $\Omega = (0, 1) \times (0, 1)$ and the right-hand side defined as a discontinuous cross-shaped function (see figure 11, left plot) written using Boolean binary functions as

$$f = (x > 1/3) * (x < 2/3) + (y > 1/3) * (y < 2/3).$$

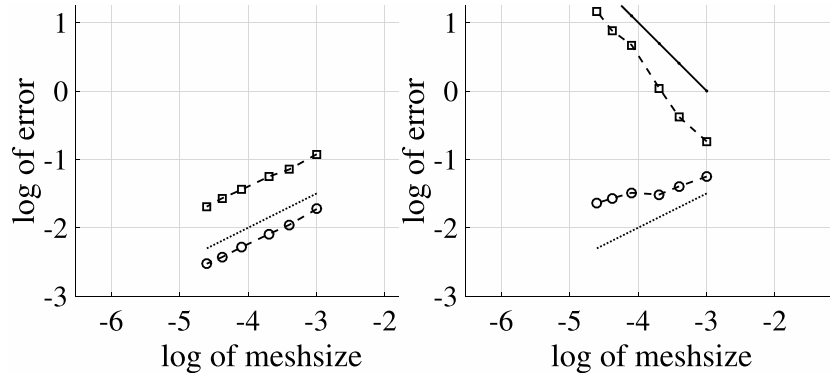


Figure 10. The convergence plot of the L^2 -norm error of the reconstruction. Circle markers denote the stabilized formulation and square markers denote the unstabilized formulation. Left: unperturbed data; right: perturbed data. The dotted line is $h^{1/2}$ and the same in both graphics; the filled line in the right plot is $0.05 h^{-1}$.

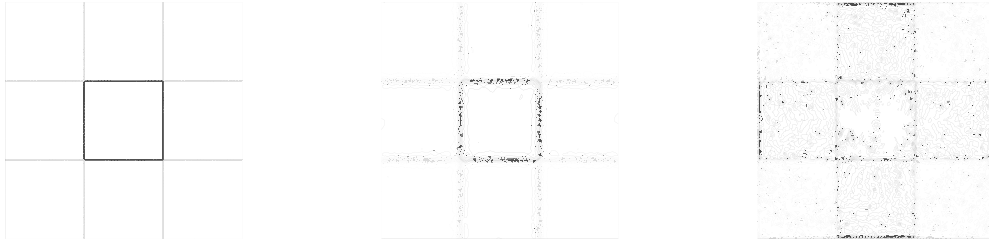


Figure 11. The contour lines of the exact and reconstructed source terms. Left: the exact source term. Middle: the reconstructed source term using stabilization and unperturbed data. Right: unstabilized reconstruction, unperturbed data.

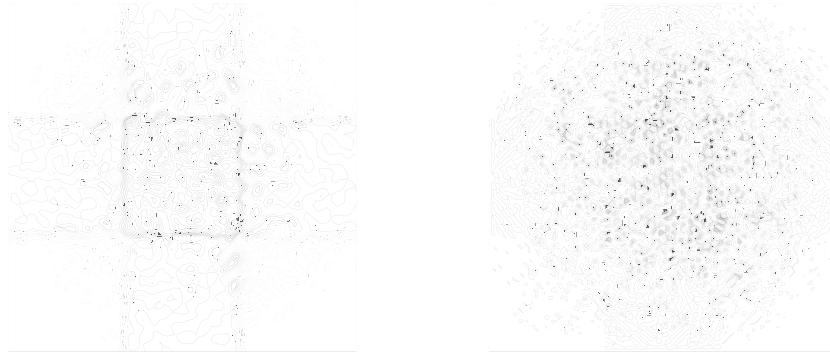


Figure 12. Left: the reconstructed source term using stabilization and perturbed data. Right: unstabilized reconstruction, perturbed data.

The data u_0 is reconstructed using P_4 finite elements on the one hand, on a mesh that is fitted to the discontinuities of f (120×120 structured), resulting in a very accurate solution ($\|u - u_h\|_{L^2(\Omega)} \leq C(120)^{-5} \|f\|_{L^2(\Omega)}$), and on the other hand on a mesh that is not fitted to the discontinuities of f (110×110 elements). The unfitted data results in spurious high-frequency oscillations with a small amplitude in the high order finite element solution, as can be seen in

figure 9 (left: fitted data and right: unfitted data). The L^2 -norm of the difference of the fitted and the unfitted solution is a good measure of the size of the perturbation. It is 1.7×10^{-4} .

First we fixed $\gamma = 10^{-6}$ after a few steps of a line search algorithm, using the same stabilization parameter for s_1 and s_5 . We solved the problem using six unstructured (Delaunay) meshes with 20, 30, 40, 60, 80 and 100 elements on the domain side. The L^2 -error in q is given in figure 10. Circle markers indicate the result obtained with the stabilized method and square markers indicate the result obtained taking $\gamma = 0$ above. In the left plot the data u_0 is given by the accurate computation and in the right plot the perturbed data is used. As can be seen in the plots, for the unperturbed data the stabilized method performs slightly better than the unstabilized method and has approximately $h^{\frac{1}{2}}$ order convergence in the L^2 -norm, which is optimal. For the perturbed data, on the other hand, the situation is dramatically different: whereas the stabilized method almost has the same convergence on coarse meshes and only stagnates on finer meshes, when the effect of the perturbation becomes important, the unstabilized method diverges.

The exact and reconstructed source function for the case of the fitted data (on the 80×80 mesh) is given in figure 11 and with unfitted data in figure 12.

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Appendix

We will give here a brief proof of the error estimates (52) for the finite element discretization of the regularized problem (12) and (13). We let Ω , be a convex polygonal domain. The optimality system (12) and (13) may formally be cast on the strong form

$$\begin{aligned} \chi_{\mathcal{M}} u_{\alpha} - \alpha \Delta u_{\alpha} - \Delta \lambda_{\alpha} &= \chi_{\mathcal{M}} u_0 \text{ in } \Omega \\ -\Delta u_{\alpha} &= f \text{ in } \Omega \\ \lambda_{\alpha} &= 0 \text{ on } \partial\Omega \\ \alpha \nabla u_{\alpha} \cdot n + \nabla \lambda_{\alpha} \cdot n &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{A.1}$$

where $\chi_{\mathcal{M}}$ denotes the characteristic function of the set \mathcal{M} . We know from (14) that $u_{\alpha} \in H^1(\Omega)$ and $\Delta u_{\alpha} \in L^2(\Omega)$ independent of α . It then follows that $\Delta \lambda \in L^2(\Omega)$. Multiplying the first equation of (A.10) with $v = -\Delta \lambda_{\alpha}$, after integration and the Cauchy–Schwarz inequality we obtain

$$\|\Delta \lambda_{\alpha}\|_{L^2(\Omega)} \leq \|\alpha \Delta u_{\alpha}\|_{L^2(\Omega)} + \|u_0 - u_{\alpha}\|_{L^2(\mathcal{M})}.$$

It then follows by elliptic regularity and (14) that

$$|\lambda_{\alpha}|_{H^2(\Omega)} \leq C(\alpha^{\frac{1}{2}} \|\nabla u\|_{L^2(\Omega)} + \|\alpha \Delta u\|_{L^2(\Omega)}). \tag{A.2}$$

Similarly, writing $y = \alpha u_{\alpha} + \lambda_{\alpha}$ we have that

$$-\Delta y = \chi_{\mathcal{M}}(u_0 - u_\alpha) \text{ in } \Omega$$

and $\nabla y \cdot n = 0$ on $\partial\Omega$. It follows that $y \in H^2(\Omega)$, with $|y|_{H^2(\Omega)} \leq C\|u_0 - u_\alpha\|_{L^2(\mathcal{M})}$, therefore, using (14) and (A.2) we have

$$\begin{aligned} |u_\alpha|_{H^2(\Omega)} &\leq \frac{1}{\alpha} |\alpha u_\alpha + \lambda_\alpha|_{H^2(\Omega)} + \frac{1}{\alpha} |\lambda_\alpha|_{H^2(\Omega)} \leq \frac{1}{\alpha} \|u - u_\alpha\|_{L^2(\mathcal{M})} + \frac{1}{\alpha} |\lambda_\alpha|_{H^2(\Omega)} \\ &\leq C(\alpha^{-\frac{1}{2}} \|\nabla u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)}). \end{aligned} \quad (\text{A.3})$$

The finite element formulation of (A.1) takes the following form: find $u_{\alpha h}, \lambda_{\alpha h} \in \mathcal{V}_h^{UC}$ such that

$$A_{UC}^\alpha[(u_{\alpha h}, \lambda_{\alpha h}), (v_h, \mu_h)] = m_\Omega(f, \mu_h) + m_{\mathcal{M}}(u_0, v_h), \quad \forall v_h, \mu_h \in \mathcal{V}_h^{UC},$$

with

$$\begin{aligned} A_{UC}^\alpha[(u_{\alpha h}, \lambda_{\alpha h}), (v_h, \mu_h)] := & m_{\mathcal{M}}(u_{\alpha h}, v_h) + \alpha a(u_{\alpha h}, v_h) \\ & + a(v_h, \lambda_{\alpha h}) + a(u_{\alpha h}, \mu_h). \end{aligned}$$

For the analysis we introduce the triple norm:

$$|||(v_h, w_h)|||_\alpha := \|v_h\|_{L^2(\mathcal{M})} + \alpha^{\frac{1}{2}} \|v_h\|_{H^1(\Omega)} + \|w_h\|_{H^1(\Omega)}. \quad (\text{A.4})$$

Using a similar argument to those leading to proposition 1, it is straightforward to show that there exists c_s such that for all $w_h, v_h \in \mathcal{V}_h^{UC}$

$$c_s |||(w_h, v_h)|||_\alpha \leq \sup_{(x_h, \mu_h) \in \mathcal{V}_h^{UC}} \frac{A_{UC}^\alpha[(w_h, v_h), (x_h, \mu_h)]}{|||(x_h, \mu_h)|||_\alpha}. \quad (\text{A.5})$$

Let $\Pi_h : H^1 \mapsto V_h$ be defined as the H^1 -projection, such that for $u \in H^1(\Omega)$, $a(u - \Pi_h u, v_h) = 0$ and $\int_\Omega (u - \Pi_h u) dx = 0$ for all $v_h \in V_h$. This projection is well posed and known to satisfy optimal error bounds in the L^2 - and H^1 -norms. Now denote the discrete errors by $\xi_h = u_{\alpha h} - \Pi_h u_\alpha$ and $\zeta_h = \lambda_{\alpha h} - i_h \lambda_\alpha$, where i_h denotes the Scott–Zhang interpolant [42]. By definition of A_{UC}^α and the orthogonality property of $\Pi_h u_\alpha$, it is easy to show that

$$\begin{aligned} A_{UC}^\alpha[(\xi_h, \zeta_h), (v_h, \mu_h)] &= m_{\mathcal{M}}(u_\alpha - \Pi_h u_\alpha, v_h) + a(v_h, \lambda_\alpha - i_h \lambda_\alpha) \\ &\leq (\|u_\alpha - \Pi_h u_\alpha\|_{L^2(\mathcal{M})} + \alpha^{-\frac{1}{2}} \|\lambda_\alpha - i_h \lambda_\alpha\|_{H^1(\Omega)}) |||(v_h, \mu_h)|||_\alpha. \end{aligned} \quad (\text{A.6})$$

Combining this result with the stability (A.5) and the standard interpolation estimates leads to the following bound for the discrete error:

$$c_s |||(\xi_h, \zeta_h)|||_\alpha \leq C(h^2 |u_\alpha|_{H^2(\Omega)} + \alpha^{-\frac{1}{2}} h |\lambda_\alpha|_{H^2(\Omega)}). \quad (\text{A.7})$$

It then follows, using a triangle inequality, standard approximation, (A.7), (A.2) and (A.3) and assuming $h^2 < \alpha < 1$, that

$$\begin{aligned} c_s |||(u_\alpha - u_{\alpha h}, \lambda_\alpha - \lambda_{\alpha h})|||_\alpha &\leq c_s (|||(u_\alpha - \Pi_h u_\alpha, \lambda_\alpha - i_h \lambda_\alpha)|||_\alpha + |||(\xi_h, \zeta_h)|||_\alpha) \\ &\leq C(\alpha^{-\frac{1}{2}} h^2 + h) \leq Ch. \end{aligned} \quad (\text{A.8})$$

As a consequence we obtain the left bound of (52)

$$\|\nabla(u_\alpha - u_{\alpha h})\|_{L^2(\Omega)} \leq C\alpha^{-\frac{1}{2}} h. \quad (\text{A.9})$$

For the L^2 -error estimate we introduce the regularized dual optimality system:

$$\begin{aligned}
\chi_{\mathcal{M}}\varphi - \alpha\Delta\varphi - \Delta\varsigma &= u_{\alpha} - u_{\alpha h} \text{ in } \Omega \\
-\Delta\varphi &= 0 \text{ in } \Omega \\
\varsigma &= 0 \text{ on } \partial\Omega \\
\alpha\nabla\varphi \cdot \mathbf{n} + \nabla\varsigma \cdot \mathbf{n} &= 0 \text{ on } \partial\Omega.
\end{aligned} \tag{A.10}$$

On the weak form the system reads

$$A_{UC}^{\alpha}[(v, w), (\varphi, \varsigma)] = m_{\Omega}(u_{\alpha} - u_{\alpha h}, v). \tag{A.11}$$

Using the weak formulation we obtain the following estimate for (A.10):

$$\|\varphi\|_{L^2(\mathcal{M})}^2 + \|\alpha^{\frac{1}{2}}\nabla\varphi\|_{L^2(\Omega)}^2 = m_{\Omega}(u_{\alpha} - u_{\alpha h}, \varphi) \leq \|u_{\alpha} - u_{\alpha h}\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)}$$

from which we conclude, using the Poincaré inequality $\|\varphi\|_{L^2(\Omega)} \leq C(\|\varphi\|_{L^2(\mathcal{M})} + \|\nabla\varphi\|_{L^2(\Omega)})$ that

$$\|\varphi\|_{L^2(\mathcal{M})} + \|\alpha^{\frac{1}{2}}\nabla\varphi\|_{L^2(\Omega)} \leq C\alpha^{-\frac{1}{2}}\|u_{\alpha} - u_{\alpha h}\|_{L^2(\Omega)}.$$

To obtain regularity estimates for φ and ς we proceed similarly as for the primal equation. Considering the first equation of (A.10), we see that since $\varsigma \in H_0^1(\Omega)$ satisfies

$$-\Delta\varsigma = u_{\alpha} - u_{\alpha h} - \chi_{\mathcal{M}}\varphi$$

there holds

$$c|\varsigma|_{H^2(\Omega)} \leq \|\Delta\varsigma\|_{L^2(\Omega)} \leq \|u_{\alpha} - u_{\alpha h}\|_{L^2(\Omega)} + \|\varphi\|_{L^2(\mathcal{M})} \leq C(1 + \alpha^{-\frac{1}{2}})\|u_{\alpha} - u_{\alpha h}\|_{L^2(\Omega)}.$$

Let $\eta = \alpha\varphi + \varsigma$, then $\eta \in H^1(\Omega)$ satisfies

$$-\Delta\eta = u_{\alpha} - u_{\alpha h} - \chi_{\mathcal{M}}\varphi$$

with homogeneous Neumann conditions and we deduce that

$$c|\eta|_{H^2(\Omega)} \leq \|u_{\alpha} - u_{\alpha h}\|_{L^2(\Omega)} + \|\varphi\|_{L^2(\mathcal{M})} \leq C(1 + \alpha^{-\frac{1}{2}})\|u_{\alpha} - u_{\alpha h}\|_{L^2(\Omega)}.$$

It then follows, using the bound on $|\varsigma|_{H^2(\Omega)}$, that

$$c(\alpha|\varphi|_{H^2(\Omega)} + |\varsigma|_{H^2(\Omega)}) \leq C(1 + \alpha^{-\frac{1}{2}})\|u_{\alpha} - u_{\alpha h}\|_{L^2(\Omega)}. \tag{A.12}$$

We are now in the position to prove the L^2 -norm error estimate. Take $v = u_{\alpha} - u_{\alpha h}$, $w = \lambda_{\alpha} - \lambda_{\alpha h}$ in (A.11) to obtain

$$\|u_{\alpha} - u_{\alpha h}\|_{L^2(\Omega)}^2 = A_{UC}^{\alpha}[(u_{\alpha} - u_{\alpha h}, \lambda_{\alpha} - \lambda_{\alpha h}), (\varphi, \varsigma)].$$

By the Galerkin orthogonality there holds

$$\begin{aligned}
A_{UC}^{\alpha}[(u_{\alpha} - u_{\alpha h}, \lambda_{\alpha} - \lambda_{\alpha h}), (\varphi, \varsigma)] &= A_{UC}^{\alpha}[(u_{\alpha} - u_{\alpha h}, \lambda_{\alpha} - \lambda_{\alpha h}), (\varphi - \Pi_h\varphi, \varsigma - i_h\varsigma)] \\
&= m_{\mathcal{M}}(u_{\alpha} - u_{\alpha h}, \varphi - \Pi_h\varphi) + \alpha a(u_{\alpha} - u_{\alpha h}, \varphi - \Pi_h\varphi) + a(\lambda_{\alpha} - \lambda_{\alpha h}, \varphi - \Pi_h\varphi) \\
&\quad + a(u_{\alpha} - u_{\alpha h}, \varsigma - i_h\varsigma) = I + II + III + IV.
\end{aligned}$$

Considering the terms of the right-hand side one by one, using (A.8) we have the standard approximation estimates for i_h , Π_h and (A.12),

$$\begin{aligned}
I &= m_{\mathcal{M}}(u_{\alpha} - u_{\alpha h}, \varphi - \Pi_h\varphi) \leq \|u_{\alpha} - u_{\alpha h}\|_{L^2(\mathcal{M})}\|\varphi - \Pi_h\varphi\|_{L^2(\Omega)} \\
&\leq Ch^3\alpha^{-\frac{3}{2}}\|u_{\alpha} - u_{\alpha h}\|_{L^2(\Omega)},
\end{aligned}$$

$$II = \alpha a(u_\alpha - u_{\alpha h}, \varphi - \Pi_h \varphi) \leq C \alpha^{\frac{1}{2}} h^2 |\varphi|_{H^2(\Omega)} \leq Ch^2 \alpha^{-1} \|u_\alpha - u_{\alpha h}\|_{L^2(\Omega)}.$$

In term *III* it is important to use the orthogonality of $\Pi_h \varphi$ in order to get the right scaling in α ,

$$\begin{aligned} III &= a(\lambda_\alpha - \lambda_{\alpha h}, \varphi - \Pi_h \varphi) = a(\lambda_\alpha - \Pi_h \lambda_\alpha, \varphi - \Pi_h \varphi) \\ &\leq Ch^2 |\lambda_\alpha|_{H^2(\Omega)} |\varphi|_{H^2(\Omega)} \leq Ch^2 \alpha^{-1} \|u_\alpha - u_{\alpha h}\|_{L^2(\Omega)}. \end{aligned}$$

Finally, we estimate term *IV* using (A.8), standard interpolation and (A.12)

$$IV = a(u_\alpha - u_{\alpha h}, \varsigma - i_h \varsigma) \leq C \alpha^{-\frac{1}{2}} h^2 |\varsigma|_{H^2(\Omega)} \leq Ch^2 \alpha^{-1} \|u_\alpha - u_{\alpha h}\|_{L^2(\Omega)}.$$

We conclude by collecting the above bounds and dividing them by $\|u_\alpha - u_{\alpha h}\|_{L^2(\Omega)}$.

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References

- [1] Albani V, De Cezaro A and Zubelli J P 2016 On the choice of the Tikhonov regularization parameter and the discretization level: a discrepancy-based strategy *Inverse Problems Imaging* **10** 1–25
- [2] Alessandrini G, Rondi L, Rosset E and Vessella S 2009 The stability for the Cauchy problem for elliptic equations *Inverse Problems* **25** 123004
- [3] Anzengruber S, Hofmann B and Mathé P 2014 Regularization properties of the sequential discrepancy principle for Tikhonov regularization in Banach spaces *Appl. Anal.* **93** 1382–400
- [4] Babuška I 1970/1971 Error-bounds for finite element method *Numer. Math.* **16** 322–33
- [5] Becker R and Vexler B 2007 Optimal control of the convection-diffusion equation using stabilized finite element methods *Numer. Math.* **106** 349–67
- [6] Becker R, Kapp H and Rannacher R 2000 Adaptive finite element methods for optimal control of partial differential equations: basic concept *SIAM J. Control Optim.* **39** 113–32
- [7] Bourgeois L and Dardé J 2010 About stability and regularization of ill-posed elliptic Cauchy problems: the case of Lipschitz domains *Appl. Anal.* **89** 1745–68
- [8] Burman E 2013 Stabilized finite element methods for nonsymmetric, noncoercive, and ill-posed problems. Part I: elliptic equations *SIAM J. Sci. Comput.* **35** A2752–80
- [9] Burman E 2014 Error estimates for stabilized finite element methods applied to ill-posed problems *C. R. Math. Acad. Sci. Paris* **352** 655–9
- [10] Burman E 2016 Stabilised finite element methods for ill-posed problems with conditional stability *Building Bridges: Connections, Challenges in Modern Approaches to Numerical Partial Differential Equations (Lecture Notes in Computational Science and Engineering vol 114)* (Cham: Springer) pp 93–127
- [11] Burman E and Ern A 2007 Continuous interior penalty *hp*-finite element methods for advection and advection-diffusion equations *Math. Comput.* **76** 1119–40
- [12] Burman E and Hansbo P 2004 Edge stabilization for Galerkin approximations of convection-diffusion-reaction problems *Comput. Methods Appl. Mech. Eng.* **193** 1437–53
- [13] Burman E and Hansbo P 2006 Edge stabilization for the generalized Stokes problem: a continuous interior penalty method *Comput. Methods Appl. Mech. Eng.* **195** 2393–410
- [14] Burman E, Hansbo P and Larson M G 2015 A stabilized cut finite element method for partial differential equations on surfaces: the Laplace–Beltrami operator *Comput. Methods Appl. Mech. Eng.* **285** 188–207
- [15] Casas E and Raymond J P 2006 Error estimates for the numerical approximation of Dirichlet boundary control for semilinear elliptic equations *SIAM J. Control Optim.* **45** 1586–611

- [16] Cheng J and Yamamoto M 2000 One new strategy for *a priori* choice of regularizing parameters in Tikhonov's regularization *Inverse Problems* **16** 31–8
- [17] de Hoop M V, Qiu L and Scherzer O 2012 Local analysis of inverse problems: Hölder stability and iterative reconstruction *Inverse Problems* **28** 045001
- [18] Deckelnick K, Günther A and Hinze M 2009 Finite element approximation of Dirichlet boundary control for elliptic PDEs on two- and three-dimensional curved domains *SIAM J. Control Optim.* **48** 2798–819
- [19] Di Pietro D A and Ern A 2012 *Mathematical Aspects of Discontinuous Galerkin Methods (Mathématiques and Applications vol 69)* (Heidelberg: Springer)
- [20] Engl H W, Hanke M and Neubauer A 1996 *Regularization of Inverse Problems (Mathematics and its Applications)* (Dordrecht: Kluwer)
- [21] Ern A and Guermond J L 2004 *Theory and Practice of Finite Elements (Applied Mathematical Sciences vol 159)* (New York: Springer)
- [22] Falk R S 1973 Approximation of a class of optimal control problems with order of convergence estimates *J. Math. Anal. Appl.* **44** 28–47
- [23] French D A and King J T 1991 Approximation of an elliptic control problem by the finite element method *Numer. Funct. Anal. Optim.* **12** 299–314
- [24] Gamallo P, Hernández E and Peters A 2011 On the error estimates for the finite element approximation of a class of boundary optimal control systems *Numer. Funct. Anal. Optim.* **32** 383–96
- [25] Gong W and Yan N 2011 Mixed finite element method for Dirichlet boundary control problem governed by elliptic PDEs *SIAM J. Control Optim.* **49** 984–1014
- [26] Griesbaum A, Kaltenbacher B and Vexler B 2008 Efficient computation of the Tikhonov regularization parameter by goal-oriented adaptive discretization *Inverse Problems* **24** 025025
- [27] Griffiths D F and Sanz-Serna J M 1986 On the scope of the method of modified equations *SIAM J. Sci. Stat. Comput.* **7** 994–1008
- [28] Groetsch C W and Neubauer A 1989 Regularization of ill-posed problems: optimal parameter choice in finite dimensions *J. Approx. Theory* **58** 184200
- [29] Hohage T and Weidling F 2015 Verification of a variational source condition for acoustic inverse medium scattering problems *Inverse Problems* **31** 075006
- [30] Ito K and Jin B 2015 *Inverse Problems: Tikhonov Theory and Algorithms (Series on Applied Mathematics vol 22)* (Hackensack, NJ: World Scientific)
- [31] Ito K and Kunisch K 1996 Augmented Lagrangian formulation of nonsmooth convex optimization in Hilbert spaces *Control of Partial Differential Equations and Applications (Laredo, 1994) (Lecture Notes in Pure and Applied Mathematics vol 174)* (New York: Dekker) pp 107–17
- [32] Jin B and Lorenz D A 2010 Heuristic parameter-choice rules for convex variational regularization based on error estimates *SIAM J. Numer. Anal.* **48** 1208–29
- [33] Kabanikhin S I 2008 Definitions and examples of inverse and ill-posed problems *J. Inverse Ill-Posed Problems* **16** 317–57
- [34] Kabanikhin S I and Schieck M 2008 Impact of conditional stability: convergence rates for general linear regularization methods *J. Inverse Ill-Posed Problems* **16** 267–82
- [35] Klibanov M V 2015 Carleman estimates for the regularization of ill-posed Cauchy problems *Appl. Numer. Math.* **94** 46–74
- [36] Kunisch K and Zou J 1998 Iterative choices of regularization parameters in linear inverse problems *Inverse Problems* **14** 1247–64
- [37] Li R, Liu W, Ma H and Tang T 2002 Adaptive finite element approximation for distributed elliptic optimal control problems *SIAM J. Control Optim.* **41** 1321–49
- [38] May S, Rannacher R and Vexler B 2013 Error analysis for a finite element approximation of elliptic Dirichlet boundary control problems *SIAM J. Control Optim.* **51** 2585–611
- [39] Nashed M Z 1987 A new approach to classification and regularization of ill-posed operator equations *Inverse and ill-Posed Problems (Sankt Wolfgang, 1986) (Notes and Reports in Mathematics in Science and Engineering)* (Boston, MA: Academic)
- [40] Neubauer A and Scherzer O 1990 Finite-dimensional approximation of Tikhonov regularized solutions of nonlinear ill-posed problems *Numer. Funct. Anal. Optim.* **11** 85–99
- [41] Roos H G, Stynes M and Tobiska L 2008 *Robust Numerical Methods for Singularly Perturbed Differential Equations (Springer Series in Computational Mathematics vol 24)* 2nd edn (Berlin: Springer) (Convection-diffusion-reaction and flow problems)

- [42] Scott L R and Zhang S 1990 Finite element interpolation of nonsmooth functions satisfying boundary conditions *Math. Comput.* **54** 483–93
- [43] Tikhonov A N, Leonov A S and Yagola A G 1998 *Nonlinear Ill-Posed Problems Vol 1, 2 (Applied Mathematics and Mathematical Computation vol 14)* (London: Chapman and Hall) (Translated from Russian)
- [44] Vexler B 2007 Finite element approximation of elliptic Dirichlet optimal control problems *Numer. Funct. Anal. Optim.* **28** 957–73
- [45] Wang L and Zou J 2010 Error estimates of finite element methods for parameter identifications in elliptic and parabolic systems *Discrete Continuous Dyn. Syst. B* **14** 1641–70